stein, Rice Institute; President E. E. Braithwaite, Western University, London, Ontario; Professor O. D. Kellogg, University of Missouri; Professor J. W. Young, Dartmouth College; Professor Tomlinson Fort, University of Alabama; and Professor F. R. Moulton, University of Chicago. After these somewhat formal exercises, the remainder of the evening was spent in social intercourse. Postcard greetings signed by those present were sent to our editor-in-chief, Professor Slaught, who was unable to be present at the meeting.

On Wednesday afternoon between the meetings of the Society and those of the Association an organ recital was given in Amasa Stone Chapel by Professor Clemens, and an opportunity was afforded for a visit to the Cleveland Art Museum. President and Mrs. Charles F. Thwing entertained the members of the Association at a tea given at their home on Thursday afternoon at four o'clock. Particular appreciation was expressed by the ladies because of the arrangements provided for their comfort and entertainment by Mrs. W. E. Beckwith and Dr. Mary F. Curtis of the College for Women of Western Reserve University; this included an automobile ride about the city on Thursday morning. Luncheon was served each day at the Case Club in the immediate vicinity of the campus; this club with the convenient lounging room supplied a means of promoting that good-fellowship which is so important a part of the Association meetings. A formal resolution was adopted at the closing session, expressing the thanks of the Association to the authorities of Western Reserve University and of Case School of Applied Science for their invitation to hold the meetings at Cleveland; to all who assisted in making the stay in Cleveland so pleasurable, in particular to President and Mrs. Thwing for their hospitality, to Professor Clemens for his interesting recital, to those who made the arrangements for the visiting ladies, and to Professors Focke, Pitcher and Wilson who as local members of the committee on arrangements planned so effectively; and finally, to the program committee who under the chairmanship of Professor C. S. Slichter prepared so successful a program.

The sessions of the Association continued through Thursday morning and afternoon, and Friday morning, Professor Cajori presiding except for the last half of the session on Thursday afternoon, when he called to the chair Professor Keppel of the University of Florida. The sessions were all held in the lecture room of the Physics Building of Case School. The program and the abstracts of the prepared papers and discussions follow. It will be seen that the topics prepared by the committee for this meeting are well adapted to the needs of the average college teacher, and, as a natural result, the papers provoked much profitable and lively discussion.

At the close of the session on Friday Professor J. W. Young gave an informal report on the activity of the National Committee on Mathematical Requirements. No account of this will be given here since full and authoritative information from Professor Young will reach our members through a formal report to be printed in the November MONTHLY.

ORDER OF TOPICS ON THE SEPARATE PROGRAM.

(1) "Undergraduate Mathematical Clubs." PROFESSOR H. E. HAWKES, Columbia University.
(2) Discussion led by PROFESSOR R. C. ARCHIBALD, Brown University, and PROFESSOR D. A. ROTHROCK, Indiana University.
(3) Presidential Retiring Address: "The Significance of Mathematics." PROFESSOR E. R. HEDRICK, University of Missouri.
(4) "Geometry for Juniors and Seniors." PROFESSOR E. B. STOUFFER, University of Kansas.
(5) Discussion led by PROFESSOR ARNOLD EMCH, University of Illinois, and PROFESSOR L. W. DOWLING, University of Wisconsin.
(6) "The Treatment of the Applications in College Courses in Mathematics." PROFESSOR L. C. PLANT, Michigan Agricultural College.
(7) Discussion led by PROFESSOR W. A. HURWITZ, Cornell University, and PROFESSOR A. M. KENYON, Purdue University.

Abstracts, numbered to correspond with the numbers on the foregoing program, are printed below, together with reports of some further informal discussions.

ABSTRACTS OF PAPERS.

(1) In speaking of undergraduate mathematical clubs Professor Hawkes first laid emphasis on the importance of such extra-curricula activities of college students as have to do with things of the mind. Although it is difficult for the average undergraduate to understand it, such activities express a deeper and more genuine loyalty to the college than athletics. After pointing out in some detail the benefits that members of a mathematical club might expect to derive from such an organization, he discussed various methods of organizing clubs. He thought that the particular type of organization made little difference provided some member of the faculty was able and willing to devote time and energy to the work of directing the students in the preparation of their papers, the outstanding fact being that the club will not run itself.

The latter part of the paper was devoted to types of topics, some of which are, and some of which are not, well adapted for presentation at the club by undergraduates.

(2) Professor Archibald gave an account of the Brown University mathematics club, whose organization, under the direction of a committee on arrangements and of a program committee, he explained. He then gave a number of facts concerning the organization of 26 undergraduate clubs whose location he indicated. The one at Smith College founded in 1899 was referred to as one of the oldest if not the oldest; and among the youngest were those founded during the past year at Alabama, North Carolina, Oklahoma, Oregon and Texas. In conclusion he spoke as follows:

"I wish to suggest that a special department of the MONTHLY be devoted to

undergraduate clubs. Probably forty clubs of this kind have been organized already and the increase in this number in the near future is sure to be rapid. There should be some connecting bond, some central source of suggestion, of interchange of ideas, and of inspiration. The MONTHLY has served as such an intermediary in the past, but it appears to me that much more might be done in the future. For instance, in each January issue might be published a list of the clubs with statistics as to membership and meetings. For future issues there would be a wealth of material to draw upon in lists of officers, programs and miscellaneous notes which might be procured from club secretaries. Then, too, such a special department would tend to draw out suggestions, discussions, papers and bibliographies for suitable program topics,—all contributing to the organization and development of these new forces, the undergraduate clubs, which are destined, I believe, in no small way to promote the cause of mathematics in America."

Professor Rothrock gave a report upon the results obtained from a questionnaire sent to 160 colleges and universities of the United States. Only a very brief summary can be given here. There were 110 replies, 31 of which reported the existence of clubs with a total membership of about 900, ranging from 10 to 106 in the different institutions. Membership is, in general, invitational, but the door is open to practically all students above freshmen who are interested in mathematics. Most clubs are conducted by students with the coöperation of the faculty. The programs consist, for the most part, of topics from the history and pedagogy of mathematics, with some attention to the curiosities of mathematics; and, as well, addresses on topics of general interest to all by faculty members and invited guests.

Thirty of the clubs responding to the questionnaire have only the highest praise in favor of the club. According to these it is highly beneficial to the students in giving them opportunity to prepare and present in a public way the results of study, and it gives them insight into many phases of mathematics not otherwise treated in college courses.

In reply to an inquiry, Professor Roe described the mathematical fraternity, Pi Mu Epsilon, at Syracuse University. This club of about 55 members, which takes the place of the usual mathematical club, is a scholarship fraternity, the membership of which is made up from the best members of the higher classes. It was asserted to be a great stimulus both to students and faculty, the students doing better work in their courses in order to qualify themselves for membership. Professor Decker added that in an institution where fraternities abound, the Greek letter mathematical fraternity appeals strongly to the students' interest.

Professors Huntington and Fort described a mathematical club at Harvard University whose membership is confined entirely to advanced students, being dependent solely on their own inspiration and guidance and independent of any embarrassing or dampening effect due to the presence and criticism of faculty members. Messrs. Barnett and Clarke described briefly the mathematical clubs at the University of Chicago, Professor Dowling and Dr. Clements those at the

University of Wisconsin, and Professors Miller and Rietz those at the University of Illinois.

A large number of members took part in this discussion by way of inquiry or suggestion. Some of these related to membership fees (ranging from a simple annual fee of twenty-five cents for postage, etc., to a fee of three dollars to provide for "spreads"), limitations of numbers, voluntary *versus* assigned papers, measures to care for the passive students and to arouse discussion in the clubs, problem solving as an integral part of college work, and still other questions. The eagerness of inquiries and the equal readiness to share experiences made it very evident that the Association, through such conferences or through some still more effective means, can be of great service in bringing to many colleges and universities the benefits to be derived from the development of mathematical clubs. A symposium of these papers and discussions will be published in the MONTHLY.

(3) The presidential address of the retiring president, Professor E. R. Hedrick, dealt with the significance of mathematics in several phases. As an offset to recent criticisms of mathematics, the essential necessity of a widespread knowledge of quantitative relations, as revealed by the existing war, was emphasized. It was urged that the Association might be a center for increasing the appreciation of mathematics of collegiate grade. Finally the recognition of applied mathematics by the Association was mentioned, and the possible function of the Association in stimulating work in applied mathematics was emphasized. The paper will be published in full in an early issue of the MONTHLY.

(4) In his paper on "Geometry for Juniors and Seniors" Professor Stouffer first discussed the need of a course in geometry for juniors and seniors, and then considered some general principles which he believed to be essential to the proper selection and arrangement of the material for such a course. A brief outline of a course in projective geometry was then given which should couple together the earlier courses in geometry and form an introduction to advanced courses which might be taken. This paper will be published at an early date in the MONTHLY.

(5) Professor Emch agreed with Professor Stouffer in the most important points. His principal postulate in his comment was that much more attention should be paid than at present to the constructive side of elementary geometrical instruction with the use of various geometrical instruments. He added that instruction in geometry should run parallel with instruction in algebra. He differs from Professor Stouffer in placing less emphasis on the synthetic as opposed to the analytic side, and in contending that the pupil in a first course in projective geometry should be enabled to solve graphically problems in perspective or central projection rather than to reason successfully on the abstract parts of the subject, the latter being reserved for a second course.

Professor Dowling found himself in substantial agreement with Professor Stouffer's paper. He objected to the vocational *raison d'être* for mathematics in general and for geometry in particular, but wished to emphasize the constructive side of projective geometry on the important ground that prospective

teachers of geometry need training in the power of visualization. He felt that the proposed outline of topics for a course in projective geometry placed the discussion of conics and the consequent construction work too far along in the course. He preferred to follow Reye in this particular, even if the removal of analytic discussion proposed by Professor Stouffer necessitated a postponement of the consideration of continuously projective forms until a later chapter. He emphasized particularly one point of the main paper in saying that the greatest cultural value of projective geometry arises from a study of the generalizing methods and principles characteristic of modern work in geometry.

In the extension of this discussion Professor Hedrick urged that geometry should be studied in this country more than is now the case; Professor Eisenhart called attention to the later volume of Darboux on the principles of analytic geometry which presents the advantages of coördinate systems, a discussion of the geometry of Cayley, and of transformations by inversion, a book which could very properly be used with juniors; Mr. Barnett mentioned a recent Italian text organized somewhat along the lines of Professor Stouffer's course; Professor P. F. Smith pointed out that later college courses in mathematics need not be justified in the way necessary with earlier courses, that the aim in projective geometry might very well be to interest the student and to prepare for his future courses; and Professor Huntington remarked that the treatment of imaginary elements must depend not on visualization but on analytic means.

(6) Professor Plant in his paper on the "Treatment of the Applications in College Courses in Mathematics" pointed out that it would first be necessary to consider why applications, other than a limited number of a geometrical character, should find any place at all in college courses in mathematics. He showed that varied applications are justified if by their use the teacher can (1) arouse an interest in the subject by bringing out its beauty through geometry, or by emphasizing its utility through mechanics; or (2) clarify the fundamental ideas that underlie the subject through geometry, physics, or mechanics; or (3) can more effectively put the student in possession of a powerful instrument for the development of the exact sciences. He also pointed out that the kind of students for whom the applications are written must be kept in mind. Students who elect courses in pure mathematics find pleasure in the logic of the treatment. The majority of students, however, belong to that class who take mathematics because it is a means to an end, and for this class applications will prove of interest only if the student is properly prepared to handle them.

After a discussion of the assumptions made by different teachers and authors concerning the preparation of the student, the conclusion was reached that it is not safe to base the treatment of applications upon ideas and principles which are likely to be outside the student's general experience. Since also instructors are quite prone to think that whatever amount of information the text may contain is sufficient for their classes, it follows that the preparation for the particular applications must be supplied by the author if he wishes to be certain that his readers are prepared to understand them. This, of necessity, reduces

the number of applications he can use and still keep his text within reasonable limits. For this reason, as well as for others, applications should be so selected and treated that they become an integral part of the course, the application bringing out more clearly certain mathematical principles and concepts and at the same time the mathematics appearing to have been created for the application.

The use of a certain kind of problem to supplement, but not to be substituted for, the applications just outlined was emphasized. Such problems may be selected from physics, chemistry, or mechanics, and are stated in the form of assumptions. Since they are so stated the student can work them, even though he does not appreciate their physical significance. While these problems serve to familiarize the student with a type of work actually encountered in the sciences, their greatest value comes from the fact that they furnish assumptions which appeal to the student's interest.

(7) [Both the speakers announced for the discussion of this paper, Professors Carver and Ling, were unable to be present because of serious illness in their families. Professors Hurwitz and Kenyon kindly consented to supply these places on the program, and their discussions, though given on brief notice, were highly appreciated.]

Professor Hurwitz, mentioning two frequently considered reasons for teaching applications in courses in mathematics, namely, the promotion of the student's interest in the subject and the clarification of mathematical principles, emphasized as an important third reason the value to cultural students of such applications as ends in themselves. He deprecated the use of many problems based on concepts neither belonging to the student's normal experience nor admitting of ready explanation to him; for such questions as rest on ideas not in the student's experience, it is advisable to state the general laws involved with brief explanation, clearly marking the distinction between the assumptions and the mathematical deductions. He warned against insistence on pseudo-rigor in the treatment of applications in elementary courses. In concluding, he pointed out the value and the practicability of requiring the students to make problems for themselves, as well as to gather them from the abundant sources of puzzle-supply in the newspapers and magazines.

The primary reason for the use of applications in college courses in mathematics, in Professor Kenyon's opinion, is to give the student an insight into the meaning and implications of the fundamental principles of mathematics. Applications treated under the guidance of this principle are a strong stimulus to the interest of the student. Professor Kenyon has scant regard for courses in which the applications rather than the mathematical principles determine the material, arrangement, and method of presentation. In case of students in scientific and technical courses, a second reason for treating applications is to develop and exercise the ability of the student to make similar applications for himself in subsequent studies and work. To do this independently and confidently requires a certain attitude and maturity of mind which results, if at all, from a somewhat extensive experience and drill in the field of mathematical and scientific ideas,

to which it is one function of the mathematics instructor to contribute as much as possible. On the other hand it is, and ought to be, hopeless to attempt to prepare the technical student for his future work by trying to show him how to make all the applications that are likely to confront him in the future.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

The Council met at five o'clock Thursday afternoon.

(1) The following nine persons and two institutions, on applications duly certified, were elected to membership.

To individual membership:

ELBERT ALLEN, Superintendent and instructor in mathematics, High School, Bainbridge, Ind.

FLORENCE E. ALLEN, Instructor in mathematics, University of Wisconsin, Madison, Wis.

CHESTER C. CAMP, Professor of mathematics, Ottawa University, Ottawa, Kan.

GUY R. CLEMENTS, Instructor in mathematics, U. S. Naval Academy, Annapolis, Md.

MARION CONNELLY, Graduate student, University of Chicago, Chicago, Ill.

JOHN H. MINNICK, Instructor in mathematics, University of Pennsylvania, Philadelphia, Pa.

SAMUEL S. PENN, Brooklyn, N. Y.

EARLE B. PHELPS, Professor of chemistry, Hygienic Laboratory, U. S. Public Health Service, Washington, D. C.

SHELTON P. SANFORD, Athens, Ga.

To institutional membership:

UNIVERSITY OF ARKANSAS, Fayetteville, Ark.

UNIVERSITY OF NEBRASKA, Lincoln, Neb.

This is the proper place to publish a list of thirteen persons and two institutions elected to membership by mail vote of the council in July, 1917:

To individual membership:

EARL J. BELCHER, Instructor in mathematics, Broaddus Institute, Philippi, W. Va.

A. C. BOSE, Deputy magistrate, Bengal provincial civil service, Simla, India.

WILLIAM E. HEAL, Efficiency accounting, Washington, D. C.

LOUIS E. MENSENKAMP, Instructor in mathematics, High School, Freeport, Ill.

VICENTE MILLS, Surveyor, Bureau of Lands, Manila, P. I.

OSCAR A. RANDOLPH, Instructor in physics, University of Colorado, Boulder, Col.

ALBERT A. RILEY, Washington, D. C.

PAULINE SPERRY, Assistant professor of mathematics, Smith College, Northampton, Mass.

H. IVAH THOMSEN, Baltimore, Md.

ROSS B. WILDERMUTH, Assistant professor of mathematics, Capital University, Columbus, Ohio.

EDNA F. WILSON, De Smet, South Dakota.

JAY W. WOODROW, Assistant professor of physics, University of Colorado, Boulder, Col.

CHIA-CHEOW YEN, Professor of mathematics, Chinese Government Engineering College, Tangshan, China.

To institutional membership:

CULVER-STOCKTON COLLEGE (formerly Christian University), Canton, Mo.
UNIVERSITY OF PORTO RICO, Mayagüez, P. R.

(2) The Council having voted at the New York meeting that the winter meeting of the Association should be held in Chicago in conjunction with the Chicago Section of the American Mathematical Society, the determination of the exact dates for this meeting was referred by the Council to the joint committee on arrangements, the composition of which, as also of the program committee, is to be announced by President Cajori after consultation with the proper authorities of the Society.

(3) Certain contemplated plans which concern the editorial care of the MONTHLY and a proposal to change the manner of choosing the secretary-treasurer are to be communicated to the whole Council and then laid before the members through the columns of the MONTHLY, in accordance with the provisions of the constitution concerning amendments to the constitution or by-laws.

(4) Measures for taking care of certain expense in the collecting of data for the National Committee on Mathematical Requirements were referred with power to the Committee on Finance of the Council.

(5) A plan for a mathematical dictionary, suggested in the main by Professor G. A. Miller, was brought by him before the Council. It was agreed that the plan, if it can be carried out, is so important and that the questions of the various details, as to size, cost, extent, etc., are so involved, that a committee of five should be appointed by the president, to make an extended study of the plan, and to report to the Council.

W. D. CAIRNS, *Secretary-Treasurer.*

ALGEBRA COURSES FOR COLLEGE JUNIORS AND SENIORS.

Edited by U. G. MITCHELL, University of Kansas.

At the fourth meeting of the Kansas Section of the Mathematical Association of America a large part of the program was devoted to the consideration of

algebra courses for college students. Students taking such courses are, for the most part, preparing to teach in secondary schools, to do research work in mathematics, or to enter applied sciences. The plan of the program committee was to have three men present the subject independently from these three different points of view and by noting the material common to the three selections proposed determine the basis for a course which would answer for all three classes of students. Summaries of the papers in the order of their presentation and of the conclusions brought out in the general discussion are given below.

A. For Students Preparing to Teach, by U. G. Mitchell, University of Kansas.

We take for granted that the college juniors and seniors referred to have taken, during the first two years of their college life, the usual freshman and sophomore courses in college algebra, plane trigonometry, analytical geometry and calculus.

The chief consideration in determining such junior-senior algebra courses is unquestionably the prospective teacher's needs which a study of algebra can supply. Laying aside, for the moment, the question as to what needs can be supplied by algebra, some of the teacher's needs which come immediately to mind are:

1. *Interpreting power.* The teacher needs to grasp quickly and accurately the meaning of a problem, whether abstract or concrete, whether stated orally or in print, and to translate this meaning into mathematical terms. This power depends largely, it is true, upon innate mentality; but it also depends largely upon familiarity with forms of statement and with the concepts and conditions involved. It is a power which may be greatly cultivated, even in persons whose mental processes are not naturally quick.

2. *Analytical Power.* By this term is meant the power to separate conditions or variables from each other and to comprehend readily their interrelations. This implies, of course, synthesizing power as well.

3. *Mathematical perspective.* The teacher needs mathematical knowledge which goes much beyond that which he has to teach in order to estimate properly the relative importance of what he teaches. For example, the teacher who has solved no equations of higher degree than the second and the teacher who is familiar with the solutions of the cubic and biquadratic and knows the essentials of the theory of elimination and the Galois theory of equations will view the quadratic equation in very different perspectives.

4. *Facility in performing algebraic operations.* Facility is here used to imply not only speed but also such mastery as combines ease of performance with accuracy.

5. *An understanding of the number system as a logical development.* The number system is first learned as a practical and not as a logical system. There is no attempt to formulate explicitly the assumptions underlying algebra in any way comparable to the recognition of the definitions, axioms and postulates of elementary geometry. Consequently students have no precise knowledge of

these assumptions or of their limitations. They readily extend to negative and imaginary numbers assumptions applicable to positive numbers only. It will be an unusually bright junior class in which there is unanimity of opinion as to the product of $\sqrt{-3}$ by $\sqrt{-5}$ or in locating the error in such simple fallacies as the "proofs" [1] that $2 = 1$ and $-1 = 1$. The teacher of elementary algebra should certainly know enough concerning the assumptions underlying algebra to be sure of his ground within the scope of his teaching. It is only by such a knowledge that he gains sufficient mastery to know definitely and exactly what operations, processes, and manipulations of symbols are valid and what are not permissible.

6. *An understanding of elementary mathematics as a historical development.* Biologists tell us that, in broad outline, the child in his development repeats the experience of his ancestry—that the theory of recapitulation obtains. Whether or not this be true in general, it is certainly true that the student who has studied successfully the historical development of mathematics gets an objective view of his own mathematical evolution, much as the study of Latin or other foreign language gives the student an objective view of English, which can be obtained in no other way. The teacher needs this view in order to understand and direct most intelligently the mathematical development of those under his instruction. It also proves an excellent source of information from which enriching and stimulating material can be drawn.

Other needs, possibly as important as some of these, might be added; but considering only these, for our present purposes, we turn to the question of what material of an algebraic nature can be used to supply these needs.

For developing *interpreting power*, problems exhibiting functional relations from any other sciences or life-conditions, graphs involving such functional relations, and problems involving pure number relations would seem to be the most suitable material. *Analytical power* can also be developed by such problems as well as by algebraic proofs.

All algebraic material would, of course, contribute to *mathematical perspective;* but the following may be selected as best adapted for the purpose: sequences, limits, theorems on limits, series (including convergence tests, exponential, binomial and logarithmic series and operations with series), permutations and combinations, multinomial theorem, symmetric functions of roots of equations, transformations of equations, solutions of cubic and biquadratic equations, systems of linear equations, determinants and elimination, simultaneous quadratics, methods of approximating roots of numerical equations (Newton's and Horner's), fundamental theorem of algebra and its corollaries.

Facility in performing algebraic operations will be aided by the work in theory of equations and can be obtained by the use of exercises of sufficient difficulty; but the probability is that juniors and seniors will have gained considerable (possibly adequate) facility of this kind from the large amount of algebraic work in the differential and integral calculus.

[1] Cf. BALL'S *Mathematical Recreations and Essays*, fourth ed., pp. 24–26.

To gain an *understanding of the number system as a logical development* the student may begin with definitions of cardinal and ordinal integers (positive, of course) and formulate exactly the laws of order and procedure for that system. The number system is next enlarged by introducing fractions on an ordinal basis and reconsidering the laws of procedure and order in relation to these new numbers. Similarly, the system is successively enlarged by the introduction of negative, irrational and complex numbers, taking care at the time of each enlargment to make such extensions, restatements and modifications of the previous laws as will apply. Approximately such a development is found, for example, in the first 75 pages of Fine's *College Algebra*.

There is, of course, no strictly algebraic material which can give an *understanding of the number-system as a historical development*. Such an understanding is best obtained from a course in the history of mathematics. If, however, no such course is given to juniors and seniors, it would seem as if some such historical material should be studied in connection with the study of the logical development of the number system.

Sufficient material has been suggested for a three-hour course throughout a year. The first semester's work might well consist of the theory of equations and related topics and the second semester's work of the logical (and somewhat historical) development of the number system, sequences, limits and series.

B. For Students Preparing to Do Research Work, by W. H. Garrett, Baker University.

It is evident that the junior-senior course in algebra will depend as to its character and extent on the course in algebra which has preceded it—the "college algebra" course which the student has taken as a freshman. I shall suppose that the junior or senior has had such a course, in which the usual topics in theory of equations and determinants have been treated in a brief and elementary way. I believe the tendency of the present day is to cut the course in algebra to the smallest dimensions possible, reserving the discussion of the more abstract parts until after the student has had courses in analytic geometry and calculus. In this connection I may say that at Harvard a student may take Mathematics 3—Introduction to Modern Algebra and Geometry—without having had a course in college algebra and so, in that course, it is necessary to take up the study of determinants from the very beginning. It is my opinion, however, that the average student will do well to have taken the college algebra before entering such a course.

Supposing, then, that the student has had a brief course in college algebra, the question remains as to what topics to present in the junior-senior course. Here, again, I find the tendency is to make the course as brief as possible. In fact, in two of the large universities, there is no mention in their catalogs of such a course being offered to undergraduates. On the other hand, more time and attention is being given to courses along geometrical lines which seem to many to be more interesting and important at that stage of the student's progress. A

three-hour course for one semester should afford ample time to cover all the necessary topics in algebra the undergraduate needs as a preparation for graduate courses and with some classes the ground could be adequately covered in two lectures a week for one semester.

In such a three-hour course there would be time for the presentation of most, if not all, of the topics named below, although not necessarily in the order mentioned. For example, there appears to be no general agreement as to whether the theory of determinants or equations should be taken up first.

Two of the topics would be complex numbers with their geometric representation and the proof of DeMoivre's theorem by mathematical induction. The proof by mathematical induction often given in the college algebra course, would better be postponed, I believe, until this time.

Under the theory of equations I should recommend about such topics and treatment as are given in Dickson's *Elementary Theory of Equations*, including the general properties of polynomials, their continuity and graphical representation, a clear statement of the fundamental theorem of algebra and the necessity for a proof, corollaries following the fundamental theorem, relation of roots to coefficients, imaginary roots, Descartes's rule of signs, elementary transformations of equations, the solution of equations by Newton's method rather than by Horner's, the algebraic solution of the cubic and quartic, multiple roots and Sturm's theorem.

I would not include the proof of the fundamental theorem of algebra. I believe it is time wasted to try to present its proof to the average junior-senior class, since any proof that can be given at this stage will seem long and difficult to the student and will not really teach him anything; whereas, if the proof is postponed to a first course in the theory of functions, it can be made very simple and instructive. In connection with the discussion of Sturm's theorem there may be need for teaching the method of finding the highest common divisor by continued division, a method which was formerly taught in every course in elementary algebra, but which is now generally relegated to the appendix or eliminated altogether.

Under optional topics might be included reciprocal equations, the elementary theory of symmetric functions and its applications to resultants and discriminants of polynomials in a single variable.

In connection with the study of determinants there is a good opportunity to do some constructive work which will be of direct use to the student later on in his graduate work. Following the definition of a determinant and some of the elementary theorems, including Laplace's development and its application to the multiplication theorem, there should be emphasized the topic of linear dependence and the application of determinants to the solution of linear equations, including the solution of systems in which the number of unknowns is not equal to the number of equations, and including systems of homogeneous and non-homogeneous equations. In this connection the first four or five chapters in Bôcher's *Introduction to Higher Algebra* present just what is needed, although the

method would probably have to be modified. The notion of rank of a determinant should be introduced as early as possible and its importance emphasized. In addition, if there is time, some work could be done in elementary invariants.

It may be said that many of the foregoing topics are valuable and interesting for their own sake, rather than as a means of preparation for higher courses in the subject, but I believe that there is need of a certain amount of such work to serve as a transition from the elementary concrete algebra of the freshman year to the abstract and highly developed algebra with its formidable notation and involved formulas which is given in the graduate school.

It is gratifying to learn from some university professors who have in their classes graduate students fresh from their college courses in mathematics, that the average first-year graduate student is not poorly prepared in algebra, but that rather it seems that this is a subject in which he is pretty well prepared and that there is no complaint to be made on this score. Perhaps not so much can be said concerning the student's preparation in geometry, and, in some colleges at least, the emphasis needs to be laid on the geometry course for juniors and seniors rather than on the algebra course.

C. For Students Preparing to Enter Applied Sciences, by A. R. Crathorne, University of Illinois.

If a student in applied science has time to take a course in algebra after the regulation courses in analytic geometry and calculus the course which would be of most benefit to him would emphasize the solution of equations, theory of probabilities, determinants and infinite series. The topics in such a course most helpful to the technical student are: General properties of equations, Descartes's rule of signs, relations between roots and coefficients, transformations of equations, algebraic solution of cubic and biquadratic equations,[1] equations whose degree exceeds four, derived functions, Rolle's theorem, separation of roots; equations in one unknown and with numerical coefficients—multiple roots, Sturm's theorem, rational roots, irrational roots, solutions by means of graphs and machines, comparison of Newton's and Horner's methods, trigonometric solutions; simultaneous equations—consistency, consistency of two equations in one unknown, equivalent equations; transcendental equations, permutations and combinations leading to the theory of probabilities and least squares, determinants, including elimination, and infinite series.

However, most students of applied science do not find a place in their course for much mathematics beyond the calculus. If they can take but one course, I think that a course in advanced calculus and differential equations emphasizing the following topics would be most useful: Differential equations, emphasis being placed on the derivation of equations, on graphical and approximate solutions and on solutions in series; definite integrals, including differentiation with respect to a parameter in the integrand or in the limits of integration; expansion

[1] These solutions of the cubic and biquadratic are not particularly useful to a technical student, but he often loses much time in finding this out. He often expects a simple solution such as he has found for the quadratic equation.

in series other than power series; partial differentiation, especially of composite functions. [For example, a clear understanding of the expression

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x},$$

where $u = f(x, y, z)$ and $z = \varphi(x, y)$.] Line integrals, with Stokes's theorem, vector analysis.

D. Conclusions.

During the general discussion the speakers modified slightly their choices of material. The following was practically agreed upon as the basis for a course suitable for all three classes of students and follows very nearly the material common to the three original selections. The order here given is not to be considered as indicating necessarily the best order of presentation: Systems of linear equations, determinants and elimination, graphical representation of polynomials, transformations of equations, solutions of the cubic and biquadratic, methods of locating and approximating roots of numerical equations—Descartes's rule of signs, Sturm's theorem, multiple roots, Horner's method of approximation, Newton's method of approximation (if only one method of approximation is given, Newton's is to be preferred), fundamental theorem of algebra and its corollaries, symmetric functions of the roots of an equation, permutations and combinations, problems involving pure number-relations.

The subject-matter above suggested would furnish about enough material for a three-hour course during one semester.

A LIST OF MATHEMATICAL BOOKS FOR SCHOOLS AND COLLEGES.[1]

Preliminary Statement. The following list contains the titles of 160 books which it is believed are suitable for purchase by the usual school or college library.[2] The plan has been carried out of forming the books into four divisions of 40 each, corresponding roughly to the mathematical attainments of the freshman, sophomore, junior and senior years, this arrangement also indicating incidentally the relative difficulty which the books present. Preference has been given throughout to books in the English language, and no text-books have been included except a few having a well-recognized value as books of reference. As it might be desirable for a school or college to know in advance the price of a

[1] Reference may be made to a similar list, but intended more especially for high schools and normal schools, which has recently been published under the auspices of the Teachers College, Columbia University, New York. The title is "A brief list of mathematical books suitable for libraries in high schools and normal schools," *Teachers College Bulletin*, Series 8, No. 3 (1916).

[2] The Library Committee was appointed at the Cambridge meeting of the Association in September, 1916, and a preliminary report was made at the New York meeting in December, 1916. (See the February Monthly, page 58.) Of the numerous tentative plans mentioned in this report, the following pages give the results of several months' work of the committee on the one considered of first importance. Editors.

book which it wished to buy, an effort has been made to ascertain the present selling price and indicate it in each case, but this has not been possible in all cases owing to the uncertainty now existing in this matter, especially as regards books published by foreign houses.

Needless to say, the list contains but a few of the possible books which might appropriately be recommended for the purpose in hand. It is hoped, however, that it indicates what may well constitute a nucleus for the college library.

THE LIBRARY COMMITTEE OF THE ASSOCIATION.

FLORIAN CAJORI, Colorado College.
E. S. CRAWLEY, University of Pennsylvania.
SOLOMON LEFSCHETZ, University of Kansas.
W. R. LONGLEY, Yale University.
R. E. ROOT, U. S. Naval Academy.
W. B. FORD, University of Michigan, *Chairman*.

BOOKS FOR FRESHMEN.

Algebra.

1. FINE (H. B.). The number-system of algebra treated theoretically and historically. Heath. $1.00.
   Admirable introduction to the critical study of algebra.
2. HALL (H. S.) and KNIGHT (S. R.). Higher algebra. Macmillan.
   Gives the technique of a wide variety of topics, with numerous examples.
3. MARSH (H. W.). Technical algebra. Wiley. $2.00.
   Shows the uses of algebra in industry.
4. SMITH (C.). A treatise on algebra. Macmillan.
   Good as a text and as a reference book. Should be in all libraries.

Geometry.

5. CASEY (J.). Sequel to Euclid. Longmans. $1.10.
   Amplifies the ordinary high school course in geometry.
6. GODFREY (C.) and SIDDONS (A. W.). Modern geometry. Cambridge University Press.
   Continues geometry in an interesting way beyond Euclid.
7. HALSTED (G. B.). Rational geometry. Wiley. $1.50.
   Considers the logical foundations of geometry.
8. MARSH (H. W.). Technical geometry. Wiley. $1.25.
   Shows the uses of geometry in modern industry.

Trigonometry and Surveying.

9. HOBSON (E. W.). Treatise on plane trigonometry. Cambridge University Press. $3.00.
   An extensive and thorough treatment of the theoretical side of trigonometry.
10. MARSH (H. W.). Technical trigonometry. Wiley. $1.50.
    Shows uses of trigonometry in industry.
11. TRACEY (J. C.). Plane surveying. Wiley. $3.00.
    Valuable both as text and field manual.

History.

12. BALL (W. W. R.). Primer of the history of mathematics. Macmillan. $0.90.
    Valuable summary of the principal events in the history of mathematics.

13. CAJORI (F.). History of elementary mathematics. Second edition. Macmillan, 1917. $1.75.
A careful work, interestingly written.

14. HILL (G. F.). The development of Arabic numerals in Europe. Oxford Press. $1.75.

15. SMITH (D. E.) and KARPINSKI (L. C.). The Hindu-Arabic numerals. Ginn. $2.00.
Scholarly presentation of the theories as to the origin of our numerals.

Miscellaneous.

16. ANDREWS (W. S.). Magic squares and cubes. Open Court Publishing Co. $1.50.
One of the few treatments of this subject in English.

17. BALL (W. W. R.). Mathematical recreations. Macmillan. $3.50.
Entertaining and instructive.

18. BRECKENRIDGE (W. E.), MERSEREAU (S. F.) and MOORE (C. F.). Shop problems in mathematics. Ginn. $1.00.
Excellent collection of applied problems in elementary mathematics.

19. BRINTON (W. C.). Graphic methods for presenting facts. Engineering News Co. $4.00.
Describes general uses of the graphic method.

20. COMPTON (A. G.). Manual of logarithmic computation. Wiley. $1.50.

21. CRACKNELL (A. G.). Practical mathematics. Longmans. $1.10.
Gives many well-graded, illustrative problems.

22. DUNCAN (R. H.). Practical curve tracing. Longmans. $1.60.

23. DUNLOP (H. C.) and JACKSON (C. S.). Slide rule notes. Longmans. $0.75.

24. FABER (A. W.). Instructions for calculating rule. Faber, Newark, N. Y.
A good manual on the slide rule.

25. GEPHART (W. F.). Principles of insurance. Macmillan. 2 vols. $1.60 each.
Elements of the mathematical aspects of insurance.

26. HENRICI (O. F. M. E.). Congruent figures. Longmans. $0.50.
Amplifies high-school geometry.

27. HINTON (C. H.). The fourth dimension. Sonnenschein, London.
Interesting treatment of this popular subject.

28. HORSBURGH (E. M.). Handbook of the exhibition of Napier relics and of books, instruments, and devices for facilitating calculation. The Royal Society of Edinburgh.

29. JOHNSON (J. F.). Practical shop mechanics and mathematics. Wiley. $1.00.

30. KOCH (E. H.). The mathematics of applied electricity. Wiley. $3.00.

31. LAGRANGE (J. L.). Lectures on elementary mathematics. Open Court Publishing Co. $1.00.
The reprint of a notable work written a century ago by a great mathematician.

32. MANNING (H. P.). Fourth dimension simply explained. Munn and Co., N. Y. $1.50.
A collection of essays on this popular subject.

33. MARSH (H. W.). Industrial mathematics. Wiley. $2.00.

34. ROW (T. S.). Geometric paper folding. Edited by BEMAN and SMITH. Open Court Publishing Co. $1.00.
Gives interesting side light on the ordinary construction problems of plane geometry.

35. SANBORN (F. B.). Mechanics problems. Engineering News Co.
One of the best collections of problems in elementary mechanics.

36. SAXELBY (F. M.). Course in practical mathematics. Longmans. $2.25.

37. SCHUBERT (H.). Mathematical recreations. Open Court Publishing Co. $0.75.

38. WELD (L. G.). A short course in the theory of determinants. Macmillan. $1.90.

39. WHITE (W. F.). Scrap-book of elementary mathematics. Open Court Publishing Co. $1.00.

40. WHITWORTH (W. A.). Choice and Chance. Deighton, Bell and Co., London.
Elementary treatment of this interesting subject, with numerous examples.

BOOKS FOR SOPHOMORES.

Analytical Geometry.

1. SALMON (G.). Conic sections. Longmans. $3.75.

2. SMITH (C.). An elementary treatise on conic sections. Macmillan. $1.40.

3. SMITH (C.). An elementary treatise on solid geometry. Macmillan. $3.25.

Calculus.

4. BYERLY (W. E.) Differential calculus. Ginn. $2.00.

5. BYERLY (W. E.). Integral calculus. Ginn. $2.00.

6. LAMB (H.). An elementary course of infinitesimal calculus. Cambridge University Press. $3.00.

7. LEIB (D. D.). Problems in the calculus. Ginn. $1.00.

8. OSGOOD (W. F.). A first course in the differential and integral calculus. Macmillan. $2.00.

9. PEIRCE (B. O.). A short table of integrals. Ginn. $1.00.

10. PERRY (J.). Calculus for engineers. Arnold. London.

Projective Geometry.

11. CREMONA (L.). Elements of projective geometry. Translation by LEUDESDORF. Oxford Clarendon Press.

12. EMCH (A.). Introduction to projective geometry and its applications. Wiley. $2.50.

13. HATTON (J. L. S.). The principles of projective geometry. Cambridge University Press.

14. MATHEWS (G. B.). Projective geometry. Longmans.

History.

15. BALL (W. W. R.). A short history of mathematics. Macmillan. $3.00.

16. CAJORI (F.). History of mathematics. Macmillan. Second edition in press.

17. FINK (C.). A brief history of mathematics. Translated by BEMAN and SMITH. Open Court Pub. Co. $1.50.

18. SMITH (D. E.). History of modern mathematics. Wiley. $1.00.

Miscellaneous.

19. BRUHNS. A new manual of logarithms to seven places of decimals. Lemcke and Buechner, New York. (1913.)
Very complete.

20. COBB (H. E.). Elements of applied mathematics. Ginn. $1.00.

21. COOLIDGE (J. L.). The elements] of non-euclidian geometry. Oxford. Clarendon Press.

22. DICKSON (L. E.). Elementary theory of equations. Wiley. $1.75.

23. DE MORGAN (A.). Elementary illustrations of the differential and integral calculus. Open Court Pub. Co. $1.00.

34. DE MORGAN (A.). A budget of paradoxes. Revised by D. E. SMITH. Open Court Pub. Co. 2 Vols. $7.00.

35. DUPUIS (N. F.). Elements of synthetic solid geometry. Macmillan. $1.60.

36. FISCHER (A.). The mathematical theory of probabilities and its application to frequency curves and statistical methods. Macmillan. $2.00.

27. KLEIN (F.). Famous problems in elementary geometry. Translated by BEMAN and SMITH. Ginn. $0.50.

28. MACFARLANE (A.). Ten British mathematicians of the nineteenth century. Wiley. $1.25.

29. MANNING (H. P.). Geometry of four dimensions. Macmillan. $2.00.

30. MELLOR (J. W.). Higher mathematics for students of chemistry and physics. Longmans. $4.50.

31. MINCHIN (G. M.) and DALE (J. B.). Mathematical drawing. Arnold. London.

32. MORITZ (R. E.). Memorabilia mathematica. Macmillan. $3.00.
Contains quotations from the writings of prominent mathematicians and anecdotes about them.

33. SCHULTZE (A.). The teaching of mathematics in secondary schools. Macmillan. $1.25.

34. SKINNER (E. B.). Mathematical theory of investment. Ginn. $2.25.

35. SMITH (C.). A treatise on algebra. Macmillan.

36. SMITH (D. E.). The teaching of elementary mathematics. Macmillan. $1.00.

37. SMITH (D. E.). The teaching of geometry. Ginn. $1.25.

38. TURNER (G. C.). Graphical methods in applied mathematics. Macmillan. $1.50.

39. WELD (L. D.). Theory of errors and least squares. Macmillan. $1.25.

40. WOLSTENHOLME (J.). Mathematical problems on subjects for the mathematical tripos examinations. 2,814 problems. Macmillan. 18 shillings.

Books for Juniors.

Geometry.

1. Durell (C. V.). A course of plane geometry for advanced students. 2 Vols. Macmillan. Vol. I, $1.50. Vol. II, $2.00.
2. Heath (T. L.). The thirteen books of Euclid's elements. Cambridge University Press. 3 Vols. $13.50.
3. Hilbert (D.). The foundations of geometry. Translation by Townsend. Open Court Pub. Co. $1.00.
4. Lobachesvki (N. I.). Theory of parallels. Translation by Halsted. Open Court Pub. Co. $1.25.
5. Milne (J. J.). Homogeneous coördinates. Cambridge University Press.
6. Milne (J. J.). Elementary treatise on cross-ratio geometry, with historical notes. Cambridge University Press.
7. Scott (C. A.). Modern analytic geometry. Macmillan.
   Now out of print, but procurable from second-hand book stores.
8. Salmon (G.). Analytic geometry of three dimensions. Revised by R. A. P. Rogers. Longmans. Vol. I (1912) $3.00. Vol. II (1915) $2.25.
9. Snyder (V.) and Sisam (C. H.). Analytic geometry of space. Holt. $2.50.

Algebra.

10. Bôcher (M.). Introduction to higher algebra. Macmillan. $1.90.
11. Burnside (W. S.) and Panton (A. W.). Theory of equations. Longmans. Vol. I $3.00. Vol. II $3.00.
12. Chrystal (G.). Textbook in algebra. 2 vols. Macmillan. $9.00.
13. Fine (H. B.). A college algebra. Ginn. $1.50.

Calculus and Differential Equations.

14. Byerly (W. E.). Problems in differential calculus. Ginn. $0.75.
15. Cohen (A.). Differential equations. Heath. $2.00.
16. Hedrick (E. R.) and Kellogg (O. D.). Applications of the calculus to mechanics. Ginn. $1.25.
17. Johnson (W. W.). A treatise on ordinary and partial differential equations. Wiley. $3.50.
18. Murray (D. A.). Differential equations. Longmans. $1.90.
19. Williamson (B.). A treatise on the differential calculus. Appleton.
20. Williamson (B.). A treatise on the integral calculus. Appleton.

Miscellaneous.

21. Boole (G.). Finite differences. Macmillan.
    Out of print, but procurable from second-hand book stores.
22. Carmichael (R. D.). Theory of numbers. Wiley. $1.00.
23. Coffin (J. G.). Vector analysis. Wiley. $2.50.
24. Dixon (A. C.). Elementary properties of elliptic functions. Macmillan.

25. ENRIQUES (F.). Problems of science. Translation by ROYCE. Open Court Pub. Co. $2.50.

26. FRANKLAND (W. B.). Theories of parallelism. Cambridge University Press. $0.90.

27. GREENHILL (A. G.). The applications of elliptic functions. Macmillan.

28. HANCOCK (H.). Elliptic integrals. Wiley. $1.25. 1917.

29. ~~KELLAND (P.) and~~ TAIT (P. G.). Dynamics of a particle. Macmillan.

30. MERRIMAN (M.) and WOODWARD (R. S.). Higher mathematics. Wiley. Five parts. $0.75 each.

31. MILLER (G. A.). Historical introduction to mathematical literature. Macmillan. $1.60. 1917.

32. MILLER (J. A.) and LILLY (S. B.). Analytic mechanics. Heath. $2.00.

33. PERRY (J.). The teaching of mathematics. Macmillan. $0.75.

34. RUSSEL (B.). Foundations of geometry. Cambridge University Press. $2.00.

35. SOMMERVILLE (D. M. Y.). Elements of non-euclidean geometry. Bell. London.

36. WILLIAMSON (B.). A treatise on dynamics. Appleton.

37. WITHERS (J. W.). Euclid's parallel postulate. Open Court Pub. Co. $1.25.

38. YOUNG (J. W.). Fundamental concepts of algebra and geometry. Macmillan. $1.60.

39. YOUNG (J. W. A.). Monographs on topics of modern mathematics. Longmans. $3.00.

40. ZIWET (A.) and FIELD (P. F.). Introduction to analytical mechanics. Macmillan. $1.60.

BOOKS FOR SENIORS AND FIRST YEAR GRADUATES.

1. BASSET (A. B.). Elementary treatise on cubic and quartic curves. Deighton, Bell and Co. Cambridge.

2. BROMWICH (T. J. I.). Theory of infinite series. Macmillan. $4.75.

3. BURKHARDT (H.). Theory of functions of a complex variable. Translation by RASOR (S. E.). Heath. $4.00.

4. BYERLY (W. E.). Harmonic functions. Wiley. $1.00.

5. BYERLY (W. E.). Fourier Series. Ginn. $3.00.

6. CANTOR (G.). Contributions to the founding of the theory of transfinite numbers. Translation by JOURDAIN. Open Court Pub. Co. $1.25.

7. CARMICHAEL (R. D.). Diophantine analysis. Wiley. $1.25.

8. COHEN (A.). The Lie theory of one parameter groups. Heath.

9. COUTURAT (L.). Algebra of logic. Translation by ROBINSON. Open Court Pub. Co. $1.50.

10. DEDEKIND (R.). On the nature and meaning of numbers. Translation by BEMAN. Open Court Pub. Co. $0.75.

11. DE MORGAN (A.). On the study and difficulties of mathematics. Open Court Pub. Co. $0.75.

12. DICKSON (L. E.). Algebraic invariants. $1.25.

13. EISENHART (L. P.). A treatise on the differential geometry of curves and surfaces. Ginn. $4.50.
14. FISKE (T. S.). Functions of a complex variable. Wiley. $1.00.
15. FORSYTH (A. R.). Differential equations. Macmillan.
16. GLENN (O. E.). The theory of invariants. Ginn. $2.75.
17. GOURSAT (E.). Course in mathematical analysis. Translation by HEDRICK (E. R.). Ginn. Vol. I. $4.00. Vol. II, two parts, $2.75 each.
18. HOBSON (E. W.). The theory of functions of a real variable. Cambridge University Press.
19. HUNTINGTON (E. V.). The continuum and other types of serial order. Second edition. Harvard University Press.
20. INGERSOLL (L. R.) and ZOBEL (O. J.). Introduction to the mathematical theory of heat conduction. Ginn. $1.60.
21. MATHEWS (G. B.). Theory of numbers—Part I. Cambridge University Press. $3.50.
22. MILLER (G. A.), BLICHFELDT (H. F.) and DICKSON (L. E.). Theory and applications of finite groups. Wiley. $4.00.
23. LONEY (S. L.). An elementary treatise on the dynamics of a particle and of rigid bodies. Cambridge University Press. G. P. Putnam's Sons. $4.00.
24. OSGOOD (W. F.). Introduction to infinite series. Harvard coöperative society. Cambridge, Mass. $0.75.
25. PEIRCE (B. O.). Newtonian potential function. Ginn. $2.50.
26. PIERPONT (J.). Theory of functions of real variables. Ginn. Vol. I, $4.50. Vol. II, $5.00.
27. POINCARÉ (H.). Science and hypothesis. Translated by HALSTED. The Science Press, New York.
28. REID (L. W.). Elements of the theory of algebraic numbers. Macmillan. $3.50.
29. SERRET (J. A.). Cours de calcul différential et intégral. Paris, Gauthier-Villars.
30. TANNERY (J.). Introduction à la théorie des fonctions d'une variable. Paris. Hermann.
31. TOWNSEND (E. J.). Functions of a complex variable. Holt. $4.00.
32. VEBLEN (O.) and LENNES (N. J.). Introduction to infinitesimal analysis. Wiley. $2.00.
33. VEBLEN (O.) and YOUNG (J. W.). Projective geometry. Vol. I. Ginn. $4.00.
34. WERTHEIM (G.). Elemente der Zahlentheorie. Leipzig. Teubner.
35. WHITEHEAD (A. N.). The axioms of projective geometry. Cambridge University Press.
36. WHITEHEAD (A. N.) An introduction to mathematics. Holt. $0.50.
37. WILSON (E. B.). Advanced calculus. Ginn. $5.00.
38. YOUNG (J. W. A.). Teaching of mathematics. Longmans. $1.50.

39. ZIZEK (F.). Statistical averages. Translation by PERSONS. Holt. $2.50.
40. ZORETTI (L.). Leçons de mathématiques générales. Paris. Gauthier-Villars.

THE DOUBLE POINTS OF RATIONAL CURVES.

By OSCAR J. PETERSON, University of Wisconsin.

Let P be a point whose homogeneous coördinates are given by the parametric equations:

$$\begin{aligned} \theta x &= f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n, \\ \theta y &= g(\lambda) \equiv b_0\lambda^n + b_1\lambda^{n-1} + \cdots + b_n, \\ \theta z &= h(\lambda) \equiv c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n, \end{aligned} \tag{1}$$

where $f(\lambda)$, $g(\lambda)$, $h(\lambda)$ have no common factor. If λ be made to vary continuously from $-\infty$ to $+\infty$, the point P will describe a continuous curve of order n. Such a curve is said to be *rational* or *unicursal*.

If two distinct values s, t of the parameter λ give the same set of coördinates, so that both determine the same point $P_{s,t}$, this point is a *double point*, or singular point of the curve. For these values s, t we have

$$f(s) : g(s) : h(s) = f(t) : g(t) : h(t),$$

or

$$\begin{aligned} g(s)h(t) - h(s)g(t) &= 0, \\ h(s)f(t) - f(s)h(t) &= 0, \\ f(s)g(t) - g(s)f(t) &= 0. \end{aligned}$$

Each of these equations is divisible by $s - t$, since it is zero for $s = t$. After division by $s - t$ let

$$\begin{aligned} F(s, t) &\equiv \frac{g(s)h(t) - h(s)g(t)}{s - t} = 0, \\ G(s, t) &\equiv \frac{h(s)f(t) - f(s)h(t)}{s - t} = 0, \\ H(s, t) &\equiv \frac{f(s)g(t) - g(s)f(t)}{s - t} = 0 \end{aligned} \tag{2}$$

be the equations. Each pair of values (s, t) which is a solution of the system of equations (2) determines a double point; and, conversely, corresponding to any double point of the curve is a pair of values (s, t) which satisfies equations (2).

(*a*) If $s \neq t$, the curve crosses itself; $P_{s,t}$ is a *node*, or *crunode*.

(*b*) If s and t are conjugate imaginaries, and the point $P_{s,t}$ is real, then $P_{s,t}$

is a *conjugate point* (isolated double point, acnode). It is the real intersection of imaginary branches of the curve; there are no other real points in its vicinity.

(*c*) If two consecutive points of the curve coincide, that is, if there is a solution (s, t) of (2) such that $s = t$, the point is a *cusp* (spinode).

In the following discussion we shall assume that all singular points which occur are either nodes, conjugate points, or cusps.

A theorem of fundamental importance in the study of rational curves is: *Every rational curve of the nth order has* $\frac{1}{2}(n-1)(n-2)$ *double points.*

This theorem was first proved by Clebsch (1864).[1] The following proof makes no use of Plücker's formulas. It follows immediately from a consideration of equations (2). Thus,

$$\begin{aligned} g(s)h(t) - h(s)g(t) \equiv\ & (st)^{n-1}[\alpha_{01}(s-t)] \\ & + (st)^{n-2}[\alpha_{02}(s^2-t^2) + \alpha_{12}(s-t)] \\ & + (st)^{n-3}[\alpha_{03}(s^3-t^3) + \alpha_{13}(s^2-t^2) + \alpha_{23}(s-t)] \\ & + \cdots + [\alpha_{0n}(s^n-t^n) + \cdots + \alpha_{n-1,n}(s-t)], \end{aligned}$$

where $\alpha_{ij} = b_ic_j - c_ib_j$.

$$\begin{aligned} F(s,t) \equiv\ & (st)^{n-1}[\alpha_{01}] + (st)^{n-2}[\alpha_{02}(s+t) + \alpha_{12}] \\ & + (st)^{n-3}[\alpha_{03}(s^2+st+t^2) + \alpha_{13}(s+t) + \alpha_{23}] \\ & + \cdots + [\alpha_{0n}(s^{n-1} + s^{n-2}t + \cdots + t^{n-1}) + \cdots + \alpha_{n-1,n}]. \end{aligned}$$

Since $s^k + s^{k-1}t + \cdots + t^k$, where k is a positive integer, can be expressed as a rational integral function of degree k in $s+t$ and st, it is possible to write $F(s,t)$ as a function of degree $n-1$ in $s+t$ and st. Similar expressions are found for $G(s,t)$ and $H(s,t)$.

$$\begin{aligned} F(s,t) &\equiv \overline{F}_{n-1}(s+t, st), \\ G(s,t) &\equiv \overline{G}_{n-1}(s+t, st), \\ H(s,t) &\equiv \overline{H}_{n-1}(s+t, st). \end{aligned}$$

By Bézout's Theorem there are $(n-1)^2$ solutions $(s+t, st)$ of $\overline{F}_{n-1} = 0$ and $\overline{G}_{n-1} = 0$. But each such pair $s+t$, st determines only one distinct pair of values s, t; there are, therefore, precisely $(n-1)^2$ solutions (s,t) of $F = 0$ and $G = 0$. All of these will also satisfy $H = 0$ except those for which both s and t are roots of $h = 0$. The n roots of $h = 0$ can be arranged in pairs in $\frac{1}{2}n(n-1)$ different ways. The number of solutions (s,t) of (2) is

$$(n-1)^2 - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}(n-1)(n-2).$$

[1] A. Clebsch, *Crelle's Journ.*, Vol. 64, pp. 43–65. See also J. C. F. Haase, *Math. Ann.*, Vol. 2 (1870), pp. 515–548; and H. Wieleitner, *Theorie der algebraischen Kurven höherer Ordnung*, pp. 74–75. The proof presented here was worked out in connection with a course in higher geometry conducted by Professor Dowling during the year 1916–17.

Hence, if δ be the number of nodes, α of conjugate points, and κ of cusps, we have

$$\delta + \alpha + \kappa = \tfrac{1}{2}(n-1)(n-2).$$

In general, not all of the $\frac{1}{2}(n-1)(n-2)$ double points can be cusps; *the upper limit of the number of cusps is* $\frac{3}{2}(n-2)$. This may be shown as follows:

In order that a point P of the curve be a cusp it is necessary and sufficient that the value of the parameter λ corresponding to the point P be a common root of the equations $F(\lambda, \lambda) = 0$, $G(\lambda, \lambda) = 0$, $H(\lambda, \lambda) = 0$. [Cf. (2), c.]

$$\begin{aligned} F(\lambda, \lambda) &\equiv h(\lambda)g'(\lambda) - g(\lambda)h'(\lambda),^1 \\ G(\lambda, \lambda) &\equiv f(\lambda)h'(\lambda) - h(\lambda)f'(\lambda), \\ H(\lambda, \lambda) &\equiv g(\lambda)f'(\lambda) - f(\lambda)g'(\lambda). \end{aligned} \tag{3}$$

We notice that

$$\begin{aligned} \rho u &= F(\lambda, \lambda), \\ \rho v &= G(\lambda, \lambda), \\ \rho w &= H(\lambda, \lambda), \end{aligned} \tag{4}$$

are the parametric equations of the curve in homogeneous line coördinates. Each of the polynomials F, G, H is of degree $2(n-1)$ in λ, the coefficient of the highest power obviously vanishing. If there are κ common roots, corresponding to κ cusps,

$$\begin{aligned} F(\lambda, \lambda) &\equiv K(\lambda)\varphi(\lambda), \\ G(\lambda, \lambda) &\equiv K(\lambda)\psi(\lambda), \\ H(\lambda, \lambda) &\equiv K(\lambda)\chi(\lambda), \end{aligned}$$

where K is of degree κ in λ, and φ, ψ, χ are of degree $2(n-1) - \kappa$. The parametric equations of the curve in line coördinates are then

$$\begin{aligned} \rho' u &= \varphi(\lambda), \\ \rho' v &= \psi(\lambda), \\ \rho' w &= \chi(\lambda). \end{aligned} \tag{4'}$$

Paralleling the preceding steps, by which we obtained (4) and (4') from (1), we can obtain the parametric equations of the curve in *point coördinates* from (4'). It is not necessary to carry this work through in detail. We must bear

[1] $$F(\lambda, \lambda) \equiv \lim_{\epsilon = 0} F(\lambda, \lambda + \epsilon) \equiv \lim_{\epsilon = 0} \frac{g(\lambda)h(\lambda + \epsilon) - h(\lambda)g(\lambda + \epsilon)}{\lambda - (\lambda + \epsilon)}$$

$$\equiv \lim_{\epsilon = 0} \frac{g[h + \epsilon h' + \frac{\epsilon^2 h''}{2} + \cdots] - h[g + \epsilon g' + \frac{\epsilon^2 g''}{2} + \cdots]}{-\epsilon}$$

$$\equiv \lim_{\epsilon = 0} [h(\lambda)g'(\lambda) - g(\lambda)h'(\lambda) + \epsilon R(\lambda, \epsilon)].$$

in mind, however, that the singularities thus brought out are not the point singularities, but the corresponding line singularities: double tangents, isolated double tangents and inflexional tangents. The equations become

$$\sigma x = \Phi(\lambda) \equiv \chi(\lambda)\psi'(\lambda) - \psi(\lambda)\chi'(\lambda),$$

$$\sigma y = \Psi(\lambda) \equiv \varphi(\lambda)\chi'(\lambda) - \chi(\lambda)\varphi'(\lambda),$$

$$\sigma z = \mathrm{X}(\lambda) \equiv \psi(\lambda)\varphi'(\lambda) - \varphi(\lambda)\psi'(\lambda).$$

If Φ, Ψ, X have no common factor, the order of the curve is $2[2(n-1) - \kappa - 1]$, this being the degree of Φ, Ψ, X; if they have ι factors in common, corresponding to ι inflexions, the order is $2[2(n-1) - \kappa - 1] - \iota$. [*Cf.* the parallel statement, (4) to (4′), above.] The order of the curve is n; hence

$$n = 2[2(n-1) - \kappa - 1] - \iota,$$

$$2\kappa + \iota = 3(n-2). \tag{5}$$

(5) indicates that the number of inflexions is odd or even according as n is odd or even; it also yields an upper limit for the number of inflexions,

$$\iota \leqq 3(n-2), \tag{6}$$

or of cusps,

$$\kappa \leqq \tfrac{3}{2}(n-2). \tag{7}$$

Wieleitner (*loc. cit.*) obtains this result (7) by means of Plücker's formulas.

When n is odd, the right-hand member of (7) is not an integer; but in this case there is at least one inflexion, and $\dfrac{3(n-2)-1}{2}$, which is the integral part of the fraction $\frac{3}{2}(n-2)$, gives the greatest possible number of cusps.

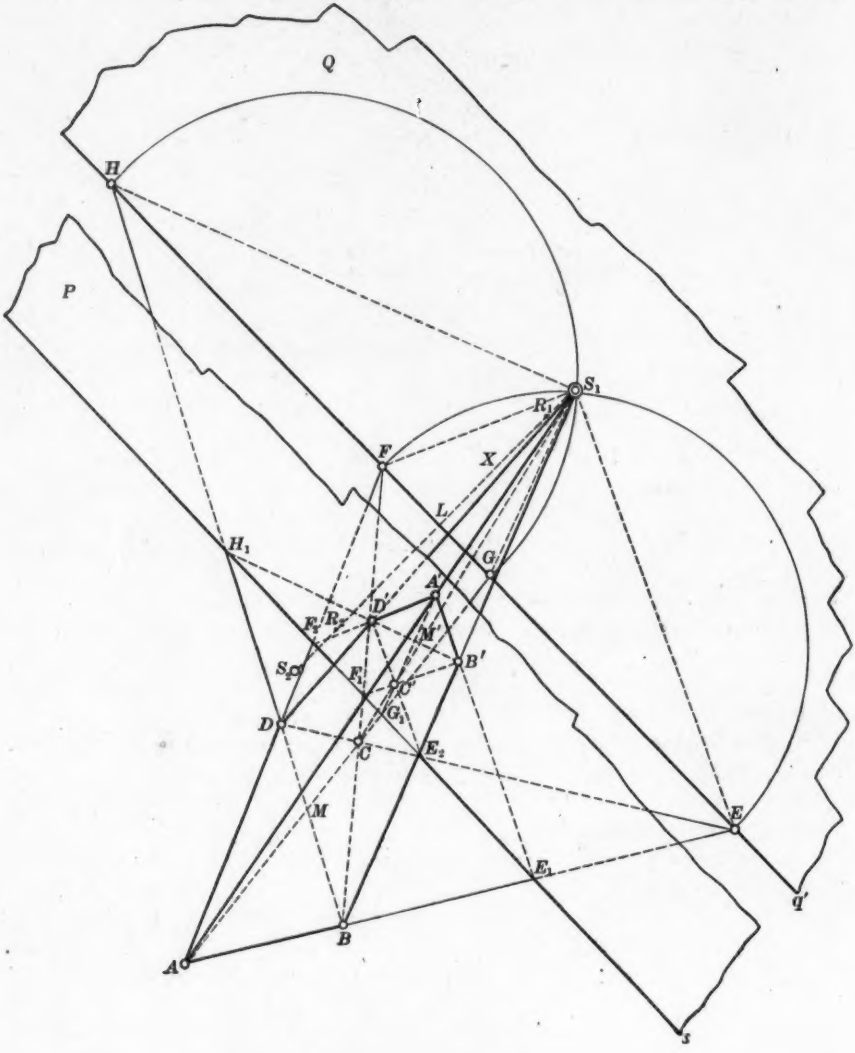
For $n > 4$, the maximum number of cusps is less than the total number of double points.

A PROBLEM IN PERSPECTIVE.

By ARNOLD EMCH, University of Illinois.

1. It is well known that any proper plane quadrangle, *i. e.*, a figure determined by four coplanar points of which no three are collinear, may be considered in an infinite number of ways as the perspective of a square, so that to the vertices of the square correspond in a certain order the vertices of the quadrangle. In such a perspective the points within the surface determined in the ordinary sense by the square and the quadrangle do not necessarily have to correspond to each other. For the sake of clearness of representation the figure shows a case of ordinary pictorial perspective in which also the ordinary surfaces correspond to each other. This does not involve loss of generality.

The purpose of the present note is to show how this problem may be treated in an extremely simple manner without going beyond the rudiments of elemen-



tary solid geometry and to emphasize, from a didactic standpoint, the importance of perspective as an introduction to projective geometry.

2. In a plane R assume any proper quadrangle $ABCD$ and let the joins of A and B, and of C and D meet at E, those of B and C, and of A and D at F.

Let q' be the join of E and F. Outside of R assume a point S_1 as the vertex of a pyramid having $ABCD$ as a base. Join S_1 to E and F; S_1E is the intersection of the planes of the faces S_1AB and S_1CD of the pyramid, S_1F that of S_1CB and S_1AD extended. Now any plane parallel to S_1E and not containing S_1 cuts the faces S_1AB and S_1CD in two parallel lines. Likewise a plane parallel to S_1F cuts S_1AD and S_1BC in two parallel lines.

Consequently every plane P parallel to the plane Q determined by S_1E and S_1F cuts the pyramid in a parallelogram $A'B'C'D'$, such that $A'B' \parallel C'D' \parallel S_1E$ and $A'D' \parallel B'C'' \parallel S_1F$. If we choose S_1 on a circle with EF as a diameter, then $S_1E \perp S_1F$, and $A'B'C'D'$ becomes a rectangle.

Let G and H be the intersections of the third pair of sides AC and BD of the quadrangle with q'. The plane determined by A, C and S_1 cuts the plane P in the join of A' and C', so that $A'C' \parallel S_1G$. Likewise, the plane determined by B, D, and S_1 cuts P in the join of B' and D' so that $B'D' \parallel S_1H$. Now a square may be defined as a rectangle with perpendicular diagonals. Hence, when S_1 also lies on a circle with GH as a diameter, so that not only $S_1E \perp S_1F$, but also $S_1G \perp S_1H$, then in the rectangle $A'B'C'D'$ the diagonals $A'C'$ and $B'D'$ are at right angles and $A'B'C'D'$ becomes a square. From this is seen that not every quadrangular pyramid admits of square sections. In order that this be the case, when the base $ABCD$ is arbitrarily given, the vertex S_1 must satisfy a certain condition, which according to the foregoing construction may be stated as follows:

Through any of three diagonals q' of the quadrangle pass any plane Q and in it draw two circles with the segment between the two diagonal points in q' as a diameter of one of the circles and with the segment determined on q' by the third pair of sides of the quadrangle as a diameter of the other circle. The intersections S_1 and S_2 of these two circles are the vertices of two pyramids with the given quadrangle as a base, such that every plane parallel to Q cuts these pyramids in squares.

In every plane Q through q' there are two such circles, and when Q revolves about q', these circles generate two spheres with their centers on q', and which intersect in a circle X, the locus of the vertices S_1 of a set (X) of pyramids with the common base $ABCD$, and admitting of square plane section. Every plane parallel to the plane determined by S_1 and q' cuts the pyramid $S_1 \cdot ABCD$ in a square.

As the complete quadrangle $ABCD$ has three diagonals, which we denote by q', r', s', there are evidently three sets of pyramids having the given quadrangle as a base and admitting of plane square sections. The loci of the vertices of these pyramids are three circles which we denote in the same order by X, Y, Z.

In all previous statements it is assumed that the three diagonals q', r', s' exist in the Euclidean sense, although, under certain modifications, they may be extended to cases in which one of the diagonals is infinitely distant, *i. e.*, in which the quadrangle is a parallelogram. The segments EF and GH on the diagonal q' and the corresponding segments on r' and s' are the diameters of the three couples of spheres that intersect in the three circles X, Y, Z. These results may be stated in the form of a theorem as follows:

THEOREM. *The locus of the vertices of all pyramids that have a proper coplanar quadrangle with three existing diagonals as a common base and which admit of square plane sections consists of three circles whose centers lie in, and whose planes are orthogonal each for each to, the three diagonals. Each of these circles is the intersection of two spheres with the two segments, like EF and GH, on each diagonal as diameters. If S_1 is a point on any of those circles attached to one of the diagonals, then any plane parallel to the plane determined by S_1 and this diagonal cuts the pyramid $S_1 \cdot ABCD$ in a square.*

3. For the sake of clearness the figure shows the construction for one diagonal, q', only. Any corresponding lines of the quadrangle and the square, like AB and $A'B'$ prolonged, meet in a point of s, the intersection of the planes R and P. The construction is still valid for the intersections R_1 and R_2 of X with the plane R. For these vertices the faces of the pyramid coincide with R. The reason for the validity in these limiting cases can easily be explained but, for the sake of brevity, will be omitted. In the language of central projection or perspective P may be considered as the plane of the object, the square $A'B'C'D'$; R as the picture-plane, S_1 as the center, s as the axis, q' as the vanishing-line of the perspective. E and F are the vanishing points of the two pairs of parallel sides of the square, and the quadrangle $ABCD$ is the perspective of the square.

4. If through S_1 we pass a plane parallel to R, it will cut P in a line t' whose perspective in a projective sense is the infinite line t of R.

By means of this perspective we can easily derive all the polar-properties of a conic from those of a circle. For instance, if we circumscribe a circle K' to the square and construct the pole T' of t' with respect to K', then T' will be projected into the pole T of the infinite line t of R with respect to the perspective K of K', which is a conic circumscribed to the quadrangle $ABCD$. By definition T is now the center of the conic K. According to the disposition in the figure this conic is an ellipse. To avoid the overloading of the figure this part of the construction, which would be simple enough, has been omitted. It follows also without difficulty that two conjugate polars of the circle through T' project into two conjugate diameters of the conic.

Thus we have reached the path by which in the historic development of projective geometry the theory of conics was derived from that of the circle.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. Longmans, Green and Co., London, 1916. xii + 179 pages. $1.50.

In this work Professor Carslaw has given an excellent introduction to the classical non-euclidean geometries in the plane, an introduction of sufficient

interest, ingenuity, and clearness to prove useful to a larger group of readers than that of "teachers of elementary geometry" for whom it was professedly written. The author has imposed on himself various limitations—not to depart from the plane; to avoid imaginary quantities, and, except in one chapter, coördinate systems; to build up a geometry independent of the principle of continuity; to avail himself of dividers (Streckenabträger) instead of compasses, wherever he could. We cannot expect such limitations to be an unmixed good for such a work. For instance, the introduction of imaginaries would have made the essential oneness of elliptic and hyperbolic geometry more apparent. There may be a question, too, as to whether it was worth while, in Chapter III, to avoid Dedekind's postulate so arduously, when Chapter IV had to use it repeatedly. This is, of course, only to say that there is no single all-surpassing way of approach to this subject.

The first chapter begins with a commentary on those axioms of Euclid, expressed or assumed, which are inconsistent with or unnecessary to the geometry to be developed. As Professor Carslaw does not set down his own fundamental assumptions, the reader takes them to be the corrected ones of Euclid. There follow proofs of a few theorems and problems common to non-euclidean and euclidean geometries.

The remainder of the first chapter, and all of the second, are historical. They tell briefly, but with a high degree of discrimination and clearness, of the work of the chief forerunners and founders of the new geometry—Saccheri, Legendre, Gauss, Schweikart, Bolyai, Lobatschevsky, Riemann.

Chapter III gives the theory of hyperbolic plane geometry, basing it on Hilbert's axiom of the existence of two parallels. First come theorems on parallels, angle of parallelism, the Saccheri (bi-rectangular isosceles) quadrilateral. From Lobatschevsky's remarkable correspondence between the five undetermined elements of a right triangle, and those of a tri-rectangular quadrilateral, Carslaw deduces a very elegant series of five associated right triangles. Points at infinity, ideal points (*i. e.*, those on and outside of the absolute) are defined as pencils of lines. The fundamental problems of construction—construction of parallel lines and of the distance corresponding to a given parallel angle—are next solved. Here for the first time compasses must replace dividers; but this does not imply a lapse into continuity, since Hilbert's axiom assures us that the circle and line whose intersection solves the problem do have a common point. The chapter ends with good discussions of the limiting curve (horocycle), the equidistant curve, and areas of triangles and polygons.

The reader who has been so incautious as to omit the preface will not expect continuity to appear unannounced in the fourth chapter (on trigonometry). Yet, on turning to p. 95, he will read: "Since we can find p to satisfy the equation $\Pi(p) = \pi/4$, there is a point Q on the Limiting Curve through P, such that the tangent at Q is parallel to the axis through P, in the opposite sense to that in which the axis is drawn." This means that the given horocycle and the equidistant curve whose distance from the axis satisfies the above equation will

intersect; and this, barring accidents, requires that the two curves be continuous. The first open avowal of continuity comes on p. 106, where we read "Further, as a increases from 0 to ∞, $f(\alpha)$ diminishes continuously from $\pi/2$ to 0." Surely Carslaw should have adopted Dedekind's postulate explicitly at the beginning of the chapter. Aside from this, the development of trigonometry, according to Liebmann, is clearly shown.

Chapter V is devoted to the discussion of three coördinate systems (cartesian, limiting curve, and polar coördinates), and to computation of elements of arc and of area in all of them. One might call all three types "polar," since all have as coördinate curves concentric circles and their radii.

Two chapters are devoted to elliptic geometry and trigonometry, the treatment being that developed by Gérard, Mansion, and Coolidge.

The last chapter, on the consistency of the non-euclidean geometries and the impossibility of proving the parallel postulate," will be the most valuable one to those who study these geometries for the sake of better understanding the logical foundation of all geometry, particularly the euclidean. The consistency of the newer systems is shown by Poincaré's incisive method of representation on the euclidean plane. Carslaw names the euclidean circles which correspond to non-euclidean lines "nominal lines," the logarithm of the cross ratio of two variable and two fixed points, the "nominal length" of the "nominal segment" between the variable points etc.—a new terminology which has the great merit of avoiding the ambiguity of the word "ideal" previously used. The book closes with quotations from Poincaré and Hadamard and remarks of the author on the "truth" of non-euclidean geometry—the latter pointing out concisely the essential difference between "geometry, the pure science" and "geometry, the applied science"; between geometry as a branch of mathematics, and as a branch of physics.

The publishers and printers have done their part well. The reviewer has his own prejudices against thick paper, and thick lines and letters in diagrams, but recognizes them to be prejudices.

A few minor faults are as follows:

Page 8. *Note.* . . . "The point C must lie between B and the pole of AB." In the elliptic plane, between any line and its pole lie all other points.

Page 46, § 25. Two lines parallel to a third are themselves parallel only if they pass through the *same* end of the third line.

Page 51. In § 28 E has two different meanings.

Page 127. The first sentence of § 74 is decidedly obscure—it seems to speak of intersecting parallels.

The following misprints were noted.

Page 55, fig. 33, for h (upper point) read h'; page 63, line 5, for III′ read 3; page 124, line 9, for $\int_0^a \int_0^{2\pi}$ read $\int_0^{2\pi} \int_0^a$; page 148, line 13, for $\frac{sc''}{Mn}$ read $\frac{sc''}{MN}$; page 149, line 12, for ABC read DBC; page 156, line 3, for 29 read 92.

Professor Carslaw has given us, probably, the best balanced elementary book on non-euclidean geometry, both its history and its theory, which has been written; for that he deserves our sincere gratitude. For the future, let us hope that, aided by further coöperation of men so instinct with life and enthusiasm as Liebmann, he can continue to enrich the mathematical world.

EDWARD S. ALLEN.

THE UNIVERSITY OF MICHIGAN.

A History of Elementary Mathematics, with Hints on Methods of Teaching. By FLORIAN CAJORI, Ph.D. Revised and Enlarged Edition. The Macmillan Company, New York and London, 1917. viii + 324 pages. $1.75.

A general demand for information in regard to the historical development of all the great sciences, coupled with the questions that have recently been raised as to the advantage of omitting a considerable part of the present teaching of mathematics, makes Professor Cajori's *History of Elementary Mathematics* a timely and welcome addition to the libraries of both teachers and students. In accordance with the title, the emphasis of the book is upon the foundations of the science of mathematics. Some advanced theories are mentioned, but puzzling details are avoided, much to the relief of the general reader, who knows that many of the upper regions in mathematics can be reached only by the climbing of stairways which are invisible until one has acquired some Aladdin's lamp of knowledge.

The historical proof of the development of mathematics as a firm and progressive science, rather than as a changing mental plaything or business calculating machine, is suggested at once in the table of contents. This short and philosophical summary furnishes to the teacher of mathematics, or to the general reader of history, the material from antiquity to modern times which enables him to place the subject where it belongs—among the old and reliable and constantly developing sciences. At practically every point which is taken up, quite full and satisfactory references are given, so that authorities on past and present historical material are placed before the reader. Those who are familiar with Professor Cajori's *History of Mathematics*, dated December, 1893, will recognize with pleasure the reappearance of many of the well-preserved mathematical antiques which he then used to exhibit clearly and forcibly the contributions of the science during the ages. The authorship and the construction of many of these antiques are again shown to be insolvable problems, but the discussion of the possible originators forms an interesting and valuable part of the new book.

Among racial contributions the book presents with great appreciation the work of the Hindus and the Arabs, and mentions the recently discovered numeral records of the Mayas of Central America as an early attempt on the part of

American races to develop mathematics; but discussion of the possibility of more important discoveries and contributions by the Japanese and Chinese than have so far been credited to them is omitted. The index includes a fairly large number of names of early mathematicians and present-day writers, but those who desire to look up such a connection as that between logic and mathematics find no assistance in the index, though the subject is discussed on p. 289; and there is also no reference to the treatment on p. 300 of the introduction of graphic methods. All these items are of interest to the teacher of the present day and should be included in the index. One wishes also that the question that is quoted so often and in so many kinds of writing as a mathematical problem of the middle ages, "How many angels can stand on the point of a needle?", had been traced to its source, in order to save the time of teachers and students in history, philosophy, literature or mathematics, who are anxious to know its origin.

The tendencies and work of the contributors to mathematics are brought out in a satisfactory way. All the old and famous mathematicians are recognized with appropriate commendations. Where space for details is limited, the lives and qualities and mathematical work of the contributors are presented in a vivid manner by a brief phrase, or even a single adjective, and make the book very readable, even to those who have usually regarded mathematics as a dry subject. Fibonaci is described as "a business man whose leisure hours were given to mathematical study," and Archimedes, "while admired by his fellow citizens for his mechanical inventions," "himself prized more highly his discoveries in pure science." The mathematician who invented a useless new method, more for a pastime than for a practical short cut, is exhibited with a proper reprimand; and the Greek mathematicians, who failed to do their share in geometry because they allowed no construction except by ruler and compasses, receive their due criticism for placing style before progress.

The lack of advance in geometry from the time of the ancient Greeks until Desargues and Pascal in the seventeenth century, and the way their discoveries were "neglected until the close of the eighteenth century" have been carefully noted, as well as the astounding fact that there was a period of "two thousand years when Egyptian mathematics was stationary." These and other points call attention to the curious delays in developments in periods when ideas were ready to flower, but were not cultivated by their owners or observed by more watchful eyes. Over and over again Professor Cajori brings out the lack of advance in theory in time to meet new and necessary demands, because attention was paid so often to arbitrary rules rather than to principles. His presentation of unsatisfactory attitudes toward arithmetic, especially in the 17th century, in the earlier part of the volume, affords considerable material for those readers who wish to take up the question outlined in his quotation from Abraham Flexner, "If, for example, only so much arithmetic is taught as people actually have occasion to use, the subject will sink into modest proportions."

The general tendency of most writers on mathematics who wish to reach modern readers is constantly to use modern notations and methods in the

elementary operations, but in this book comparisons of past and present methods are made possible by the use of ancient styles in discussing some of the fundamental processes. This enables the author to emphasize the pressure of principles as a factor in leading to new discoveries and improved notation, even though the forms in which these principles were first presented pass away, as in his summary of the difficulties due to the present notation in dealing with limits and irrationals. It seems possible that the remark of Professor Cajori, p. 236, "In case of radicals, what a gain it would be, if we could burst the chains which tie us to the past," may furnish the pressure which is needed to do away with many handicaps in notation, especially for the introduction of the metric system to complete our decimal furniture, or the addition of graphic representation in elementary mathematics.

All through the volume the influence of the needs of practical life is emphasized, and case after case is given which shows how great advances in theory have come from those peoples, and from those periods, wherein the demand led solitary mathematicians and busy teachers to give extra time and thought to methods of training the next generation and to seeking help of every sort. The biological sources are indicated when the quinary, decimal, and duodecimal number systems are called "the three systems based on human anatomy." The fields where late cultivation has brought out new results are illustrated by notice of the modern work on triangles. The way in which long and unfruitful efforts to solve a practical problem have finally led to the addition of a previously unknown number characteristic is shown in the study of the history of the quadrature of the circle. Through such discussions Professor Cajori opposes the attitude of certain philosophers who "would make us believe that observation is not needed or developed in mathematical study," and helps the average person to realize the fact that even a young mathematical student may bring to light a peculiarly valuable short cut or make a worthy discovery.

On the whole, the book contains an amount of material and suggestions that is likely to lead many readers to agree with the conclusion which Professor Cajori states in his last sentence, "Certain it is that the teaching of mathematics will require constant modifications in order to keep pace with the ever changing needs of modern life."

ROXANA H. VIVIAN.

WELLESLEY COLLEGE.

PROBLEMS AND SOLUTIONS.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

ALGEBRA.

487. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Show in two ways that 0.5623 is not a root of $(1 - m)e^m = 2e^{-1}$, e being the Napierian base. Find the correct value of m, and test the result in two ways.

488. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} = 0,$$

where

$$a_k = \frac{\sin(k\theta + \alpha)}{\sin k\theta},$$

and θ and α have any values that do not make a denominator zero.

GEOMETRY.

520. Proposed by ALBERT A. BENNETT, University of Texas.

On a given tangent to a circle determine a point such that, if a secant be drawn joining this point to the extremity of the diameter which is perpendicular to the given tangent, the segment of this secant exterior to the circle will be equal in length to a given segment.

521. Proposed by R. M. MATHEWS, Riverside, California.

A variable circle, with center on the line l and passing through a fixed point P, cuts a fixed circle in A and B. Prove that the common chord AB and the perpendicular to l through P intersect in a fixed point.

CALCULUS.

435. Proposed by B. F. FINKEL, Drury College.

Show that

$$\int_0^\infty e^{-x^2-(a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation, rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, pages 106–107.

436. Proposed by ARTEMAS MARTIN, LL.D., Washington, D. C.

A circle of radius a is drawn at random on a circular slate of radius r. If another circle of radius a be drawn on the slate, what is the probability that the second circle will intersect the first?

MECHANICS.

352. Proposed by C. N. SCHMALL, New York City.

A glass rod is balanced partly in and partly out of a cylindrical tumbler, with its lower end resting against the vertical wall of the tumbler. If ϕ and ψ are the maximum and minimum angles, respectively, which the rod can make with the vertical plane, and θ is the angle of friction, show that

$$\theta = \tfrac{1}{2}\tan^{-1}\left(\frac{\sin^3\phi - \sin^3\psi}{\sin^2\phi\cos\phi + \sin^2\psi\cos\psi}\right).$$

353. Proposed by CLIFFORD N. MILLS, Brookings, North Dakota.

A uniform beam of oak, 10 feet long, 15 inches deep and 10 inches wide, sustains, in addition to its own weight, a load of 5,000 lbs. placed at the center. Find the greatest bending moment and the greatest stress in the fibers. Take the specific gravity of oak as .934.

NUMBER THEORY.

270. Proposed by GERSHOM N. CARMICHAEL, Urbana, Ill.

Does there exist a fraction p/q in its lowest terms such that the ratio of the sum of the divisors of p to the sum of the divisors of q is equal to p/q? Give a method of finding such fractions not in their lowest terms.

271. Proposed by HORACE OLSON, Chicago, Ill.

Prove that if x, y, z, u, v, and w are integers such that $x^2 + y^2 = u^2$, $x^2 + z^2 = v^2$, $y^2 + z^2 = w^2$, then the product $xyzuvw$ is divisible by 518400.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

474. Proposed by A. A. BENNETT, University of Texas.

Show that the value of the infinite continued fraction, all of whose coefficients are unity,

$$1 + \frac{1}{1+}\,\frac{1}{1+}\cdots, \quad \text{is} \quad \tfrac{1}{2}(1+\sqrt{5}).$$

Also find an explicit algebraic formula for the nth convergent.

SOLUTION BY C. C. YEN, Tangshan, North China.

1. Let x denote the value of the continued fraction. Then,

$$x = 1 + 1/x, \quad \text{i. e.,} \quad x^2 - x - 1 = 0,$$

therefore,

$$x = \tfrac{1}{2}(1 + \sqrt{5}),$$

where the sign for the radical is positive, since x is evidently positive.

2. Let p_n/q_n denote the nth convergent. Then

$$p_1 = 1, \quad q_1 = 1, \quad p_2 = 2, \quad q_2 = 1. \tag{I}$$

Also, since all the coefficients are unity,

$$p_n = p_{n-1} + p_{n-2}, \quad q_n = q_{n-1} + q_{n-2} \quad (n = 3, 4, 5, \cdots). \tag{II}$$

And, it follows, therefore, from I and II,

$$q_n = p_{n-1} \quad (n = 2, 3, 4, \cdots). \tag{III}$$

Now consider the series

$$1 + x + 2x^2 + p_3x^3 + p_4x^4 + \cdots + p_nx^n + \cdots,$$

where the coefficients p_n satisfy II. It is a recurring series whose scale of relation is $1 - x - x^2$, and whose generating function is $1/(1 - x - x^2)$. Hence, p_n is the coefficient of x^n of the expansion of this function; and, by III, q_n is the coefficient of x^{n-1} of the same expansion.

If we put $\alpha = -\tfrac{1}{2}(1 + \sqrt{5})$, $\beta = -\tfrac{1}{2}(1 - \sqrt{5})$, we get

$$\frac{1}{1 - x - x^2} = \frac{1}{\alpha - \beta}\left(\frac{1}{\alpha - x} - \frac{1}{\beta - x}\right),$$

whence the general term of the expansion of $1/(1 - x - x^2)$ is

$$\frac{1}{\alpha - \beta}\left\{\left(\frac{1}{\alpha}\right)^{n+1} - \left(\frac{1}{\beta}\right)^{n+1}\right\}x^n = \frac{1}{\alpha - \beta}\left\{\frac{\beta^{n+1} - \alpha^{n+1}}{(\alpha\beta)^{n+1}}\right\}x^n.$$

But $\alpha\beta = -1$, and $q_n = p_{n-1}$. It follows, therefore, that

$$p_n = (-)^n\left\{\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right\}, \qquad q_n = (-1)^{n-1}\left\{\frac{\alpha^n - \beta^n}{\alpha - \beta}\right\}.$$

Hence, finally, the nth convergent is given by

$$-(\alpha^{n+1} - \beta^{n+1})/(\alpha^n - \beta^n),$$

where α, β are the roots of the equation $x^2 - x - 1 = 0$.

Also solved by PAUL CAPRON, N. P. PANDYA, and O. S. ADAMS.

475. Proposed by E. B. ESCOTT, Kansas City, Mo.

A man makes a contract to purchase a house, making a cash payment down and agreeing to make monthly payments of a dollars, interest being charged at 6 per cent., the balance of the monthly payments being credited on the principal. Find a formula for M_n, the balance due after n payments.

SOLUTION BY C. R. DUNCAN, Amherst, Massachusetts.

Let M_0 = balance due after the original cash payment, and r = rate of interest per month (= .06/12), then M_0r = interest due at end of first month, and $a - M_0r$ = amount paid back on the principal M_0.

At the end of the second month the interest would be less, the difference being equal to the interest on the amount paid back the first month, or $(a - M_0r)r$. But as all monthly payments are to be equal this amount would be credited on the principal. Hence, the amount paid back on the principal at the end of the second month is

$$(a - M_0r) + (a - M_0r)r = (a - M_0r)(1 + r).$$

Similarly, the amount paid back at the end of the third month would be equal to the amount paid back the second month plus the difference in interest between the second and third months, or

$$(a - M_0r)(1 + r) + (a - M_0r)(1 + r)r = (a - M_0r)(1 + r)^2.$$

Hence, the amount paid back at the end of the nth month $= (a - M_0r)(1 + r)^{n-1}$. Therefore, the total amount paid back in n months is

$$(a - M_0r) + (a - M_0r)(1 + r) + (a - M_0r)(1 + r)^2 + \cdots + (a - M_0r)(1 + r)^{n-1} = \frac{(a - M_0r)[(1 + r)^n - 1]}{r},$$

and the balance due after n payments is

$$M_n = M_0 - \frac{(a - M_0r)[(1 + r)^n - 1]}{r}.$$

Putting the right-hand member equal to 0 and solving for a, we have a formula for finding the amount of the monthly payments required to pay back the principal M_0 in a given number of months,

$$a = \frac{M_0r(1 + r)^n}{(1 + r)^n - 1}.$$

Also solved by G. W. HARTWELL, E. J. OGLESBY, HORACE OLSON, A. R. NAUER, PAUL CAPRON, G. PAASWELL, H. N. CARLETON, J. B. REYNOLDS, and the PROPOSER.

476. Proposed by W. HAROLD WILSON, University of Illinois.

Prove that if $x_h \neq x_j$ $\qquad (h, j = 1, 2, \cdots, n, h \neq j)$,

then

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{h=1}^{n}{}' (x_i - x_h)} = 1,$$

where the prime indicates the omission of zero factors in the denominator.

II. Solution by the Proposer.

It is convenient to use the notation

$$\Delta = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{h,j=1}^{n} (x_h - x_j), \qquad h < j.^1$$

It is also convenient to designate the minor of Δ with respect to x_i^{n-1} by Δ_i.

$$\Delta_i = \prod_{h,j=1}^{n}{}' (x_h - x_j), \qquad h < j,$$

where the prime indicates that no factor containing x_i enters. It now follows at once that

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{h=1}^{n}{}' (x_i - x_h)} = \sum_{i=1}^{n} \frac{x_i^{n-1}\Delta_i}{(-1)^{i-1} \prod_{h=1}^{i-1} (x_h - x_i) \prod_{h=i+1}^{n} (x_i - x_h) \prod_{h,j=1}^{n}{}' (x_h - x_j)}$$

$$= \sum_{i=1}^{n} \frac{(-1)^{i-1} x_i^{n-1}\Delta_i}{\prod_{h,j=1}^{n} (x_h - x_j),\ h < j} = \frac{1}{\Delta} \sum_{i=1}^{n} (-1)^{i-1} x_i^{n-1}\Delta_i = \frac{1}{\Delta}\Delta = 1. \qquad h < j,$$

Also solved by A. A. Bennett, G. W. Hartwell, and E. H. Worthington.

GEOMETRY.

508. Proposed by J. E. Rowe, State College, Pa.

The trilinear coördinates of the Brocard triangle are $(s_3^2, s_1s_2s_3, s_1^3)$, $(s_2^3, s_1^3, s_1s_2s_3)$, and $(s_1s_2s_3, s_3^3, s_2^3)$ where s_i $(i = 1, 2, 3)$ are the sines of the angles of the fundamental triangle. Show that the Brocard triangle and the fundamental triangle are in perspective, and that the trilinear coördinates of the center of perspectivity are s_i^{-3} $(i = 1, 2, 3)$ instead of s_i^3 which are incorrectly given in Clebsch *Vorlesungen über Geometrie*, p. 323.

Solution by William Hoover, Columbus, Ohio.

Use of homogeneous coördinates permits us to replace s_1, s_2, s_3 by a, b, c and we have

$$(0, 2\Delta/b, 0), \quad (c^2, abc, a^2); \qquad (0, 0, 2\Delta/c), \quad (b^2, a^2, abc); \qquad (2\Delta/a, 0, 0), \quad (abc, c^2, b^2)$$

for the pairs of sets of trilinear coördinates of the corresponding vertices of the fundamental and Brocard triangles.

The equation of a straight line passing through two points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ is

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{1}$$

Thus the straight lines joining the pairs of vertices are

$$a^2\alpha - c^2\gamma = 0 \quad (2), \qquad -a^2\alpha + b^2\beta = 0 \quad (3), \qquad b^2\beta - c^2\gamma = 0 \quad (4).$$

The three lines

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \quad (5), \qquad l_2\alpha + m_2\beta + n_2\gamma = 0 \quad (6), \qquad l_3\alpha + m_3\beta + n_3\gamma = 0 \quad (7)$$

are concurrent if

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0. \tag{8}$$

(2), (3), and (4) are, therefore, concurrent and the fundamental and Brocard triangles are in perspective.

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[1] L. G. Weld, *The Theory of Determinants*, Art. 89, p. 169.

The proportional values of α, β, γ in (5) and (6) are given by

$$\frac{\alpha'}{m_1n_2 - m_2n_1} = \frac{\beta'}{n_1l_2 - n_2l_1} = \frac{\gamma'}{l_1m_2 - l_2m_1} \tag{9}$$

and those in (2) and (3) by

$$\frac{\alpha'}{b^2c^2} = \frac{\beta'}{a^2c^2} = \frac{\gamma'}{a^2b^2} \quad (10), \qquad \text{or} \qquad \alpha' \Big/ \frac{1}{a^2} = \beta' \Big/ \frac{1}{b^2} = \gamma' \Big/ \frac{1}{c^2} \quad (11),$$

satisfying (4) and proving the theorem.

The fundamental triangle is in perspective with two other Brocard triangles, the centers of perspective being $1/b^2$, $1/c^2$, $1/a^2$; $1/c^2$, $1/a^2$, $1/b^2$.

Also solved by J. W. CLAWSON, C. P. SOUSLEY, and S. W. REAVES.

509. Proposed by NORMAN ANNING, Chilliwack, B. C.

A picture whose coördinates are (0, 0), (50, 0), (50, 50), and (0, 50) is repeated on a smaller scale as part of itself with the coördinates (7, 0), (31, 7), (24, 31), (0, 24). Locate the vanishing point.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let P, P_1, P_2, $\cdots$ denote the picture and its successive images, and let O_1, O_2, O_3, $\cdots$ be the successive images of the first vertex. We shall find the vanishing point by determining the limiting position of O_n as n increases indefinitely.

It readily follows from the data of the problem that the side of P_1 is one half that of P and makes with it an angle $\theta = \tan^{-1} 7/24$. It is clear that the side of P_n bears the same relations to the side of P_{n-1} for all values of n.

By projecting the successive segments OO_1, O_1O_2, O_2O_3, $\cdots$ on the coördinate axes, it readily follows that the coördinates of the limiting position of O_n are given by the following infinite series:

$$x = 7 + \frac{7}{2}\cos\theta + \frac{7}{2^2}\cos 2\theta + \cdots + \frac{7}{2^n}\cos n\theta + \cdots,$$

$$y = \frac{7}{2}\sin\theta + \frac{7}{2^2}\sin 2\theta + \cdots + \frac{7}{2^n}\sin n\theta + \cdots.$$

Multiplying the second equation through by $i = \sqrt{-1}$, adding the result to the first equation, denoting $\cos\theta + i\sin\theta$ by v, and remembering that $\cos n\theta + i\sin n\theta$ is by De Moivre's formula equal to v^n, we have

$$x + iy = 7\left[1 + \frac{v}{2} + \left(\frac{v}{2}\right)^2 + \cdots + \left(\frac{v}{2}\right)^n + \cdots\right].$$

Using the formula for the sum of a geometric progression and then replacing v by its value

$$\frac{24}{25} + i\frac{7}{25},$$

we have

$$x + iy = \frac{364}{29} + i\frac{98}{29}.$$

Hence, the coördinates of the required vanishing point are (364/29, 98/29).

Also solved by J. B. REYNOLDS and W. R. RANSOM.

CALCULUS.

424. Proposed by OSCAR S. ADAMS, Washington, D. C.

What is the value of

$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}?$$

SOLUTION BY EDWARD H. WORTHINGTON, University of Pennsylvania.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx.$$

Hence,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1 \qquad \text{and} \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\log x dx,$$

since differentiation may be carried under the sign.

Since

$$\log x = \int_0^\infty \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha}d\alpha, \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\int_0^\infty \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha}d\alpha dx.$$

$$\therefore \quad \Gamma'(1) = \int_0^\infty \int_0^\infty e^{-x}\frac{e^{-\alpha} - e^{-\alpha x}}{\alpha}dxd\alpha = \Gamma(1)\int_0^\infty \left(e^{-\alpha} - \frac{1}{1+\alpha}\right)\frac{d\alpha}{\alpha}.$$

(1) Hence, $\dfrac{\Gamma'(1)}{\Gamma(1)} = \displaystyle\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)}\right)\frac{d\alpha}{\alpha}$; also $\dfrac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \displaystyle\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)^{1/2}}\right)\frac{d\alpha}{\alpha}$

(2) and $$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \int_0^\infty \left(-\frac{1}{1+\alpha} + \frac{1}{(1+\alpha)^{1/2}}\right)\frac{d\alpha}{\alpha}.$$

Changing $1 + \alpha$ to $1/\alpha$, (2) becomes, if the primes are dropped, and convergency considered,

$$\lim_{\epsilon=0}\left[\int_{1-\epsilon}^0 \frac{1-\alpha^{-1/2}}{1-\alpha}d\alpha = \int_{1-\epsilon}^0 \frac{d\alpha}{-\alpha^{1/2}(\alpha^{1/2}+1)} = \int_{-\epsilon}^0 -\alpha^{-1/2}[1 - \alpha^{1/2} + \alpha - \alpha^{3/2} + \alpha^2 \cdots]d\alpha\right]$$

$$= +2[1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} \cdots] = +2\log_e 2.$$

Since $\log_e 2 = 0.69315$

$$\therefore \quad \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = +1.38630.$$

Also solved by G. PAASWELL, H. M. TERRILL, and V. M. SPUNAR.

NUMBER THEORY.

254. Proposed by HORACE OLSON, Chicago, Ill.

Find three integers x, y, z, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all perfect squares.

SOLUTION BY V. M. SPUNAR, Chicago, Ill.

Let

$$x^2 + y^2 = \square = a^2, \quad (1) \qquad\qquad y^2 + z^2 = \square = c^2, \quad (3)$$

$$x^2 + z^2 = \square = b^2, \quad (2) \qquad\qquad x^2 + y^2 + z^2 = \square = d^2. \quad (4)$$

A complete solution of (4) is as follows:

$$x = m^2 + n^2 - p^2 - q^2, \quad y = 2(mp + nq), \quad z = 2(mq - np), \quad d = m^2 + n^2 + p^2 + q^2. \quad (5)$$

First we remember that one of the two integers A and B satisfying the relation, $A^2 + B^2 = \square$, must be even.

Next, suppose y and z even, then (3) shows that c is even, and after removing the common factor 4 we find again that either $(y/2)^2$, or $(z/2)^2$ is a multiple of 4. But from (5) it is obvious that if x be odd three and only three of the numbers m, n, p, q must be odd or three even. This leads, however, to y and simultaneously z having the same factor 2, which, after suppressing in (3), leads to the conclusion again, that one number must be even, which is impossible.

Hence, the proposition is impossible.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

RELATING TO REQUIRED MATHEMATICS FOR WOMEN STUDENTS.

By EMILIE N. MARTIN, Mount Holyoke College.

In recent months I have been present at many informal discussions in regard to the required work of a college course. The background of such discussions has been the woman's college, and the aim of all the participants has been to find the best method of training the woman student for her part in the work of the world.

Some argue that with the large amount of required work in college a student has too little time in which to sample new courses or to devote herself to a subject for which she shows an aptitude. Their demand is: "Lessen the required work." Others argue that the required work of a college course is seldom more than sufficient to give the solid foundation of general culture upon which the structure of specialized subjects may be safely and permanently reared.

Some of the arguments of the opponents of required mathematics have a curiously familiar sound. To be sure they are not put as crudely as in the world outside the college. There no one hesitates to ask: "What is the use of mathematics for a girl? She will never need more than the elements of arithmetic." The opponents of the subject in a woman's college, especially if they are themselves women, are somewhat wary of asking that time-honored question. They know that they are supposed to claim the same stiff intellectual training for women as for men, so they avoid stating the question in bald terms of sex. Instead they say: "Every student who enters our college has already had several years of mathematics. She has enough for all practical purposes. [This is nothing but the old query revised.] She has already found out whether or not she has an aptitude for the subject. If she cares for it, by all means let her elect it, but if she has no aptitude for the subject, let her take something that is really worth her while instead of wasting her time on a subject she will never use."

My belief is that in spite of her school training, however thorough, the student needs some college mathematics if she is to have an education that will send her out into life with the best general equipment. Mathematics as taught in college is viewed from an angle different from that used in the school-room. This statement does not apply to solid geometry which is only an extension of the plane geometry of the school, and which we hope some day to see put back in its proper place in connection with plane geometry. Take, however, college algebra and trigonometry. Both of these subjects make use of material in the way of ideas and methods that the student has already worked with in school, but this material is handled in a very different way. In her algebra the school-girl is concerned almost entirely with processes. She needs but little theory.

This little is sometimes explained to her, sometimes she has to remember and reproduce it; but even in the latter case the average student seems to have acquired little grasp of the underlying principles. The consequence is that almost every college freshman has a definite idea of logical reasoning as connected with the subject of geometry, but has little idea of it in connection with any other part of mathematics.

In college the freshman, while once more dealing with symbols and processes that were familiar in school, is now concerned with them from the standpoint of logical combinations. She is applying to them the methods that she had thought confined to geometry, deducing the laws that rule in the subject of her study, and expressing these laws in exact mathematical language. Unless much emphasis is laid upon this side of mathematics in the freshman classes, the claims of its enemies would seem to me to have some foundation. The application of the theory to special problems is necessary as making the subject more vivid and as preparing the student for the application of abstract reasoning to all manner of problems stated in advance; but the true value of the course lies in the training it gives in applying logical processes to the mathematical concepts already familiar from school days.

The average student may not like to find that the mechanical work of her school algebra is replaced in college by this demand upon her reasoning powers, yet she is ordinarily able to master the required amount of freshman mathematics in spite of the extra difficulty. On the other hand there are always a few students to whom such a treatment of mathematics offers an insuperable difficulty,—students who seem to have no idea of logical sequence. In their statement of a geometrical construction no attention is paid to the logical order in which the lines must be drawn, in their statement of a proof the effect precedes the cause, and their minds are so constructed that it seems impossible to convince them of error—one order seems to them as good as another.

This latter class of students is always cited by our opponents as the strongest argument for dispensing with a general requirement of mathematics. "Why torture such a student," they say, "with a subject for which she has no fitness? Why should she not take the subjects for which she shows some aptitude, and let mathematics severely alone?" They usually omit to name the college subjects that do not require at least some modicum of reasoning ability. My feeling, on the contrary, is that no matter what other subjects such a student has to omit, mathematics is one subject absolutely essential to her training. Such a student needs to develop her reasoning powers, and freshman mathematics gives her the best field for practice. Other subjects also require logical ability, but often the logical framework is so obscured by the newness of the material, the unfamiliarity of the nomenclature, and the large number of strange concepts, that the value of the subject from the standpoint of logical training is quite lost.

As for the examples that the opposers of required mathematics quote from time to time of women brilliant in other lines of college work who found themselves totally unable to do even enough work in freshman mathematics to gain a

passing grade, I cannot deny that such women may exist, but I wish to record my deep conviction that the majority of these cases have been diagnosed incorrectly. I have seen so many students of mediocre abilities fight their way through the difficulties of required mathematics by sheer common-sense and will power, that I am sure that most of the cases cited so solemnly are cases of "I will not" and not of "I cannot." Because they did not want to master a difficult subject, they were willing to profess incompetence in order to get their own way.

Here the question of sex enters in. Parents and guardians who would suffer keen mortification if the boy for whose education they are responsible were in danger of being rejected by his college because of his failure in required mathematics will condone any shortcomings of the girl in that line with a deprecating, "You know that one does not think so much of a failure in mathematics for a girl." But if sex must be considered in this matter, why not consider it from this other standpoint, namely, that the woman is prone to look at everything from the personal side? Her own feelings and her background, or lack of background, color the medium through which she views such subjects as history and literature, and affect her judgments of the facts. The personality of her instructor is a factor in inclining her either to believe or disbelieve his interpretations of the theories of economics or philosophy. Of course there are fundamental laws in all these subjects so well established that the personal element cannot enter into their consideration, but there is also a large body of conclusions from these laws, and it is these conclusions from laws that are sometimes only partially understood that are now in question. Herein lies a great advantage of mathematics; it furnishes the woman student with a subject in which the validity of the conclusions drawn from its laws can easily be tested, and in which the personality of the instructor and the bias of the student can play no part.

The foes of the mathematics requirement then say: "Suppose we grant that our students need training in reasoning, and that as women they need especially training in reasoning upon an impersonal subject. Why not then require a course in a science that shall be the equivalent of the mathematics course in these two respects? Any science gives a student the opportunity to acquire the habit of assembling data, rejecting the extraneous elements, and forming the fitting conclusion. No one can bring any personal prejudice into the interpretation of the phenomena that would be discussed in a required course in any established science." Such a suggestion seems quite to overlook the fact that the only sciences that furnish a training at all equivalent to that of mathematics are those that have mathematics as their foundation. Without a preliminary training in mathematics that almost necessitates the inclusion of trigonometry the most rigorous of sciences can only be treated from the more or less popular point of view. The training so acquired is no doubt valuable, but it cannot satisfactorily replace the required work in mathematics.

Another argument that is often brought against this college requirement is that it gives a subject in which the students have already had several years' experience an unfair advantage over others that are not begun until after college

is entered, for each year of required work in a familiar subject postpones the opportunity for them to become acquainted with new subjects in which they may later desire to specialize. This argument does not take into account the fact of which I have already spoken, that even in freshman mathematics the methods are used that must be used in any further study of the subject, so making a break between the character of the work in school and that of the work in college. There are sometimes freshmen whose work in all school studies has been so good that no one subject seems to stand out as more particularly suited to them than another, who first realize that mathematics is the subject that they want for special work when they become acquainted with it afresh in their required work. Without this requirement these students will probably be lost to the subject. They may of course elect mathematics, but, under the false impression that the rather mechanical methods of mathematical work already familiar to them are to be continued in college, they usually prefer entering upon some new subject. Given the recognized attractions of the new and untried, mathematics, if not required, will be at a disadvantage as compared with other subjects.

In addition to this unfortunate effect upon the department of mathematics there are other deleterious effects upon both school and college that seem to me bound to follow the dropping of mathematics from our required work. With the best of intentions on the part of the high-school teachers the grade of work is lower in a subject that is necessary for entrance to college but that is not to be tested in the college class room than it is in a subject that will be so tested. In this I speak from experience. I once saw the effect upon the preparation in algebra, when the college with which I was connected eliminated college algebra from the required work, retaining solid geometry and trigonometry. The school work in geometry was kept up to the standard, for that work was expected to have its class-room test. The college work in trigonometry, however, revealed very clearly the fact that, while the required subjects of algebra had been nominally studied in school, the work had been done very superficially. Furthermore there will be a disastrous change in the quality of the teaching, in so far as it is done by women, if more and more women go out to teach in the schools who have had no mathematics beyond that of their high school course. The lack of background is at least as serious a fault in the teaching of mathematics as in the teaching of any other subject.

In the college itself there will probably be a marked effect upon the science departments. If the student is permitted an unfettered choice of her required science—and no one seems to hesitate a moment as to the necessity of such a requirement—the non-mathematical sciences (if such sciences truly exist) will be overrun with more students than can be handled easily, while the sciences known as mathematical will have only the few students who have brought from their school days a love of mathematics and no dread of its symbols. If the curriculum committee meets this situation by requiring the choice of one of the mathematical sciences, even then there will be a difficulty because of the poor equipment

in mathematics possessed by the students. Under such circumstances the science requiring the least amount of mathematical knowledge will naturally be given the preference, and the instructors in even the most mathematical of the sciences will have to confine themselves to a rather popular treatment, if they wish to have a fair proportion of the students elect their subject. In any case much of the value of these sciences as aids to exact thinking will be lost. Of course, for any further work in the selected science some mathematics is necessary, and under the supposed arrangement the student must take in her maturer years in college the fundamental work that she now acquires in her freshman year.

With the two apparently contradictory tendencies at present noticeable—one, to minimize for women even in the science courses in college the necessity of any mathematical training beyond that of the high school course; the other, to encourage these same students to place more and more emphasis upon their work in science, especially in the line of laboratory research,—it is evident that the majority of women workers in science will soon be forced to limit themselves to those fields in science that can be cultivated by means of the very simplest mathematical tools. These fields may be wide and they may be fertile, but by permitting this limitation women are denying to themselves the equality of opportunity with men that has been won for them at such a cost by the pioneers in the struggle for the right of women to share in the higher education.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Mr. CLYDE T. LEVY, a member of the Association, died at his home, Clinton, Mo., on July 23, 1917.

Assistant Professor E. I. SHEPARD, of Williams College, has entered the military service as captain in the Officers' Reserve Corps.

Mr. J. L. RILEY, of the State Normal School at Tahlequah, Oklahoma, has been appointed professor of mathematics in the Junior College at Stephenville, Texas.

Mr. J. B. ROSENBACH, engineer of maintenance of the Atchison, Topeka and Santa Fe railway, has resigned to become instructor in mathematics at the University of New Mexico.

Mr. C. A. STANWICK, an electrical engineer and member of the Association, has been commissioned second lieutenant in the Engineers' Officers Reserve Corps and called into active service.

Mr. C. A. EPPERSON, associate professor of mathematics at the First District Normal School, Kirksville, Mo., has been granted leave of absence and is in military training at Fort Sheridan.

Professor L. K. ADKINS, head of the department of mathematics at the Wisconsin State Normal School, La Crosse, Wis., has been appointed first lieutenant in the Regular Army, and ordered to France with General Pershing.

Dr. E. A. KIRCHER, Benjamin Peirce instructor at Harvard University, has been commissioned a captain in the coast artillery and is located at Fort Strong, Boston, Mass.

Mrs. MQYME LOGSDON, of Hastings College, Nebraska, has been appointed to an instructorship in mathematics at Northwestern University.

At the University of Maine, Dr. NORBERT WIENER has resigned to enter the military service, and Associate Professor H. R. WILLARD has been appointed statistician under Mr. HOOVER and reported at Washington in September.

Assistant Professor P. L. THORNE, of New York University, is in the military training camp at Plattsburg; and instructor H. H. PRIDE has completed the training at the Madison Camp and has been commissioned second lieutenant in the new National Army.

Assistant Professor J. A. NYSWANDER, of the University of Nevada, has been granted leave of absence to serve in the National Army. The vacancy in the department of mathematics was filled by the appointment of Mrs. NYSWANDER to serve during her husband's absence.

Three instructors of mathematics in the University of Illinois have resigned to assume other positions. Dr. F. W. REED has entered the aviation service; Dr. LEVI WILSON has joined the mathematical staff of the U. S. Naval Academy; and Dr. W. W. DENTON has been appointed associate professor of mathematics at the Worcester Polytechnic Institute.

Mr. VOLNEY WELLS, instructor in mathematics at the University of Michigan, has accepted a position in the University of Pittsburgh; instructor J. W. BALDWIN has been elected to a position in the mathematical department of the Michigan State Normal College, Ypsilanti; and Dr. A. L. MILLER, instructor in mathematics, has resigned to enter business.

At Cornell University, Dr. J. V. MCKELVEY, senior instructor in mathematics, has been granted leave of absence, and after serving in the training camp at Madison Barracks, has been commissioned a second lieutenant in the National Army; Dr. R. E. GILMAN, instructor in mathematics, has also been granted leave of absence and has been commissioned captain in the Coast Artillery.

The twenty-fourth summer meeting of the American Mathematical Society was held at Western Reserve University and the Case School of Applied Science, Cleveland, Ohio, on September 4–6, 1917. The attendance at this meeting was unusually large. Nineteen titles of papers appeared upon the printed program and several others not previously announced were presented. The readers represented twelve colleges and universities of the United States. The last session, Thursday forenoon, was in conjunction with the Mathematical Association of America, at which Professor L. P. EISENHART, of Princeton University, gave an address on "Darboux's contribution to geometry." The Society joined with the Association in a dinner at the Hotel Statler on Wednesday evening.

At the University of Chicago the following graduate students have received appointments not previously announced: Mr. G. H. CRESSE, instructor in mathematics at the University of Michigan; Mr. FRANK M. WEIDA, instructor in mathematics at the University of Iowa; Mr. O. W. ALBERT, instructor in mathematics at Grinnell College, Iowa; Dr. FLORA LE STOURGEON, instructor in mathematics at the LIGGETT School, Detroit, Mich., Dr. W. P. OTT, Dr. L. S. SHIVELY, and Dr. H. F. MCATEE, who received their doctorates at the June convocation, return to their former positions at Vanderbilt University, Mt. Morris College, Ill., and William Jewell College, Mo., respectively.

The following promotions of members of the Association have not been previously announced in these columns: LOUIS BRAND and C. N. MOORE to full professorships and J. H. KINDLE and E. S. SMITH to assistant professorships at the University of Cincinnati; E. B. STOUFFER to an associate professorship at the University of Kansas; L. C. MATHEWSON and C. R. DINES to assistant professorships at Dartmouth College; and A. C. LUNN to an associate professorship at the University of Chicago.

The following new appointments have been reported:

Dr. R. B. ROBINS, of Yale University, instructor in mathematics at the University of Michigan; Mr. I. L. MILLER, of Indiana University, professor of mathematics at Carthage College, Carthage, Illinois; Miss MARION E. STARK, graduate fellow in Brown University, professor of mathematics at Meredith College, Raleigh, N. C.; Dr. C. C. CRUMP, of Carleton College, assistant professor of astronomy and director of the Perkins Observatory of Ohio Wesleyan University; Mr. VERN JAMES, of Indiana University, instructor in mathematics at the Carnegie Institute of Technology, Pittsburgh, Pa.

The report of the Library Committee of the Association, which appears in this issue, will, it is hoped, furnish a concrete answer to the numerous inquiries that have been received by the MONTHLY concerning appropriate lists of books for college libraries. It is proposed to provide reprints of this report which may be had at cost by application to the secretary, W. D. CAIRNS, 27 King Street, Oberlin, Ohio. If requests for copies could be made at once it would greatly facilitate the determination of how large an edition to publish.

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THE SIGNIFICANCE OF MATHEMATICS.*

By E. R. HEDRICK, University of Missouri.

Several circumstances combine to render peculiarly fitting a consideration at this time of the significance of mathematics. Of late we have heard much from real or alleged educators, tending to show a lack of appreciation on their part, if not on the part of the public, of the vital role which mathematics plays in the affairs of humanity. These attacks were beginning to receive some hearing in the educational world, on account of their reiteration and their vehemence, if not through intrinsic merit.

A counter influence of tremendous public force, whose import is as yet seen only by those most nearly interested, has now arisen through the existence of war and the necessities of war. To the layman, lately told by pedagogical orators that mathematics lacks useful applications, the evident need of mathematical training on every hand now comes as a distinct surprise.

The attacks on mathematics, and the lay conception of the entire subject, center naturally around elementary and secondary instruction. We ourselves, college teachers of mathematics, have commonly talked of current practice and of reforms largely with respect to secondary education. The third influence which contributes toward the present situation and which may strongly affect its future development is the formation and the existence of this great Association, which affords for the first time in the history of America an adequate forum for the discussion of the problems of collegiate instruction in mathematics.

As retiring president of the Association, I know of no more fitting topic than that which I have chosen. It vitally concerns us; it is bound up with the functions of this Association; and the times in which we live seem to point forcibly toward its consideration. I shall attempt to outline to you my own views on

* Retiring Presidential Address delivered at the second summer meeting of the Mathematical Association of America, at Cleveland, Ohio, September 6, 1917.